Class: B. Tech (Unit I)

I have taken all course materials for Unit I from Book Introduction to Electrodynamics by David J. Griffith (Prentice- Hall of India Private limited).

Students can download this book form given web address;

Web Address : **https://b-ok.cc/book/5103011/55c730**

All topics of unit I (vector calculus & Electrodynamics) have been taken from **Chapter 1, Chapter 7 & Chapter 8** from said book (https://b-ok.cc/book/5103011/55c730). I am sending pdf file of Chapter 1 Chapter 7 & chapter 8.

Unit-I: Vector Calculus & Electrodynamics: (8 Hours)

Gradient, Divergence, curl and their physical significance. Laplacian in rectangular, cylindrical and spherical coordinates, vector integration, line, surface and volume integrals of vector fields, Gauss-divergence theorem, Stoke's theorem and Green Theorem of vectors. Maxwell equations, electromagnetic wave in free space and its solution in one dimension, energy and momentum of electromagnetic wave, Poynting vector, Problems.

CHAPTER

1 Vector Analysis

1.1 ■ VECTOR ALGEBRA

1.1.1 ■ Vector Operations

If you walk 4 miles due north and then 3 miles due east (Fig. 1.1), you will have gone a total of 7 miles, but you're *not* 7 miles from where you set out—you're only 5. We need an arithmetic to describe quantities like this, which evidently do not add in the ordinary way. The reason they don't, of course, is that **displacements** (straight line segments going from one point to another) have *direction* as well as *magnitude* (length), and it is essential to take both into account when you combine them. Such objects are called **vectors**: velocity, acceleration, force and momentum are other examples. By contrast, quantities that have magnitude but no direction are called **scalars**: examples include mass, charge, density, and temperature.

I shall use **boldface** (**A**, **B**, and so on) for vectors and ordinary type for scalars. The magnitude of a vector **A** is written |**A**| or, more simply, A. In diagrams, vectors are denoted by arrows: the length of the arrow is proportional to the magnitude of the vector, and the arrowhead indicates its direction. *Minus* **A** (−**A**) is a vector with the same magnitude as **A** but of opposite direction (Fig. 1.2). Note that vectors have magnitude and direction but *not location:* a displacement of 4 miles due north from Washington is represented by the same vector as a displacement 4 miles north from Baltimore (neglecting, of course, the curvature of the earth). On a diagram, therefore, you can slide the arrow around at will, as long as you don't change its length or direction.

We define four vector operations: addition and three kinds of multiplication.

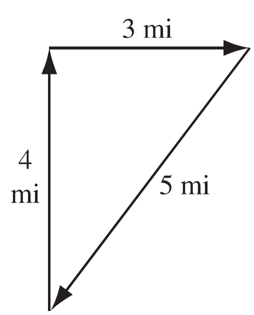


FIGURE 1.1

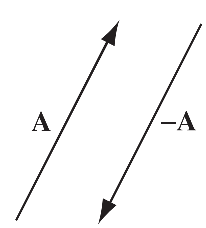


FIGURE 1.2

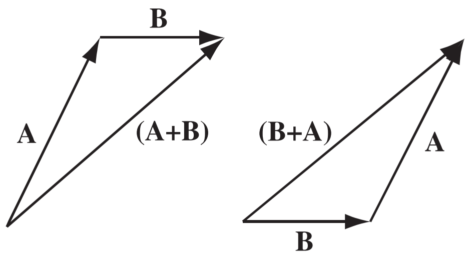


FIGURE 1.3

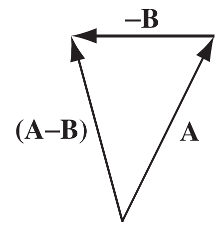


FIGURE 1.4

(i) Addition of two vectors. Place the tail of $\mathbf{B}$ at the head of $\mathbf{A}$; the sum, $\mathbf{A}+\mathbf{B}$, is the vector from the tail of $\mathbf{A}$ to the head of $\mathbf{B}$ (Fig. 1.3). (This rule generalizes the obvious procedure for combining two displacements.) Addition is *commutative:*

$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A};$$

3 miles east followed by 4 miles north gets you to the same place as 4 miles north followed by 3 miles east. Addition is also *associative:*

$$(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C}).$$

To subtract a vector, add its opposite (Fig. 1.4):

$$\mathbf{A}-\mathbf{B}=\mathbf{A}+(-\mathbf{B}).$$

(ii) Multiplication by a scalar. Multiplication of a vector by a positive scalar a multiplies the *magnitude* but leaves the direction unchanged (Fig. 1.5). (If a is negative, the direction is reversed.) Scalar multiplication is *distributive:*

$$a(\mathbf{A}+\mathbf{B})=a\mathbf{A}+a\mathbf{B}.$$

(iii) Dot product of two vectors. The dot product of two vectors is defined by

$$\mathbf{A}\cdot\mathbf{B}\equiv AB\cos\theta, \tag{1.1}$$

where θ is the angle they form when placed tail-to-tail (Fig. 1.6). Note that $\mathbf{A}\cdot\mathbf{B}$ is itself a *scalar* (hence the alternative name **scalar product**). The dot product is *commutative,*

$$\mathbf{A}\cdot\mathbf{B}=\mathbf{B}\cdot\mathbf{A},$$

and *distributive,*

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C})=\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}. \tag{1.2}$$

Geometrically, $\mathbf{A}\cdot\mathbf{B}$ is the product of A times the projection of $\mathbf{B}$ along $\mathbf{A}$ (or the product of B times the projection of $\mathbf{A}$ along $\mathbf{B}$). If the two vectors are parallel, then $\mathbf{A}\cdot\mathbf{B}=AB$. In particular, for any vector $\mathbf{A}$,

$$\mathbf{A}\cdot\mathbf{A}=A^2. \tag{1.3}$$

If $\mathbf{A}$ and $\mathbf{B}$ are perpendicular, then $\mathbf{A}\cdot\mathbf{B}=0$.

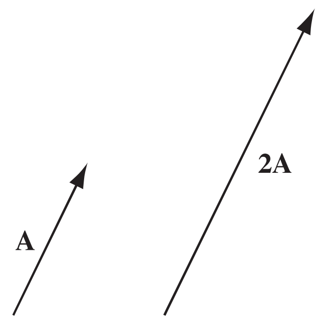


FIGURE 1.5

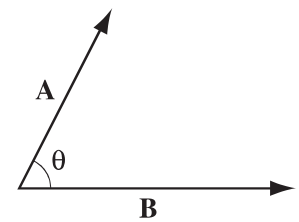


FIGURE 1.6

Example 1.1. Let $\mathbf{C} = \mathbf{A} - \mathbf{B}$ (Fig. 1.7), and calculate the dot product of $\mathbf{C}$ with itself.

Solution

$$\mathbf{C} \cdot \mathbf{C} = (\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) = \mathbf{A} \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{B} - \mathbf{B} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{B},$$

or

$$C^2 = A^2 + B^2 - 2AB \cos\theta.$$

This is the **law of cosines**.

(iv) Cross product of two vectors. The cross product of two vectors is defined by

$$\mathbf{A} \times \mathbf{B} \equiv AB \sin\theta \, \hat{\mathbf{n}}, \tag{1.4}$$

where $\hat{\mathbf{n}}$ is a **unit vector** (vector of magnitude 1) pointing perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$. (I shall use a hat (ˆ) to denote unit vectors.) Of course, there are *two* directions perpendicular to any plane: "in" and "out." The ambiguity is resolved by the **right-hand rule**: let your fingers point in the direction of the first vector and curl around (via the smaller angle) toward the second; then your thumb indicates the direction of $\hat{\mathbf{n}}$. (In Fig. 1.8, $\mathbf{A} \times \mathbf{B}$ points *into* the page; $\mathbf{B} \times \mathbf{A}$ points *out* of the page.) Note that $\mathbf{A} \times \mathbf{B}$ is itself a *vector* (hence the alternative name **vector product**). The cross product is *distributive,*

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C}), \tag{1.5}$$

but *not commutative.* In fact,

$$(\mathbf{B} \times \mathbf{A}) = -(\mathbf{A} \times \mathbf{B}). \tag{1.6}$$

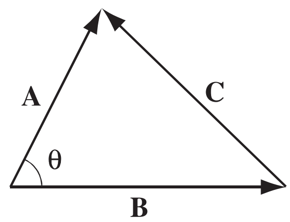


FIGURE 1.7

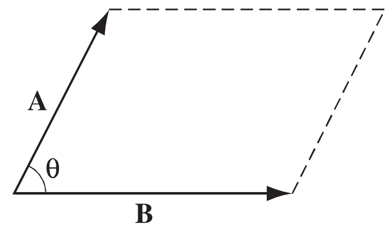


FIGURE 1.8

Geometrically, $|\mathbf{A} \times \mathbf{B}|$ is the area of the parallelogram generated by **A** and **B** (Fig. 1.8). If two vectors are parallel, their cross product is zero. In particular,

$$\mathbf{A} \times \mathbf{A} = \mathbf{0}$$

for any vector **A**. (Here **0** is the **zero vector**, with magnitude 0.)

Problem 1.1 Using the definitions in Eqs. 1.1 and 1.4, and appropriate diagrams, show that the dot product and cross product are distributive,

a) when the three vectors are coplanar;

! b) in the general case.

Problem 1.2 Is the cross product associative?

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \stackrel{?}{=} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}).$$

If so, *prove* it; if not, provide a counterexample (the simpler the better).

1.1.2 ■ Vector Algebra: Component Form

In the previous section, I defined the four vector operations (addition, scalar multiplication, dot product, and cross product) in "abstract" form—that is, without reference to any particular coordinate system. In practice, it is often easier to set up Cartesian coordinates x, y, z and work with vector **components**. Let $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ be unit vectors parallel to the x, y, and z axes, respectively (Fig. 1.9(a)). An arbitrary vector **A** can be expanded in terms of these basis vectors (Fig. 1.9(b)):

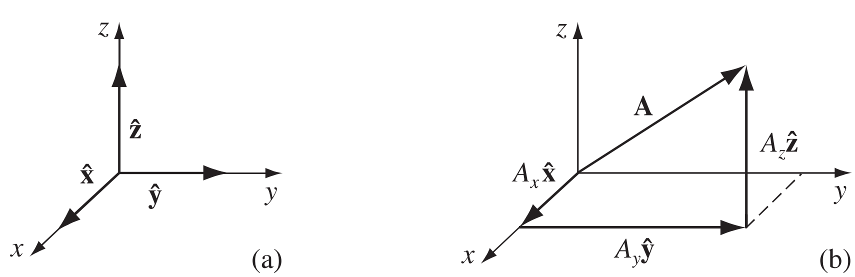


FIGURE 1.9

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}.$$

The numbers A_x, A_y, and A_z, are the "components" of **A**; geometrically, they are the projections of **A** along the three coordinate axes ($A_x = \mathbf{A}\cdot\hat{\mathbf{x}}$, $A_y = \mathbf{A}\cdot\hat{\mathbf{y}}$, $A_z = \mathbf{A}\cdot\hat{\mathbf{z}}$). We can now reformulate each of the four vector operations as a rule for manipulating components:

$$\begin{aligned}\mathbf{A}+\mathbf{B} &= (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) + (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}})\\ &= (A_x + B_x)\hat{\mathbf{x}} + (A_y + B_y)\hat{\mathbf{y}} + (A_z + B_z)\hat{\mathbf{z}}.\end{aligned} \tag{1.7}$$

Rule (i): *To add vectors, add like components.*

$$a\mathbf{A} = (aA_x)\hat{\mathbf{x}} + (aA_y)\hat{\mathbf{y}} + (aA_z)\hat{\mathbf{z}}. \tag{1.8}$$

Rule (ii): *To multiply by a scalar, multiply each component.*

Because $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are mutually perpendicular unit vectors,

$$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1; \quad \hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{x}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = 0. \tag{1.9}$$

Accordingly,

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \cdot (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}})\\ &= A_xB_x + A_yB_y + A_zB_z.\end{aligned} \tag{1.10}$$

Rule (iii): *To calculate the dot product, multiply like components, and add.*
In particular,

$$\mathbf{A}\cdot\mathbf{A} = A_x^2 + A_y^2 + A_z^2,$$

so

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}. \tag{1.11}$$

(This is, if you like, the three-dimensional generalization of the Pythagorean theorem.)

Similarly,[1]

$$\begin{aligned}\hat{\mathbf{x}}\times\hat{\mathbf{x}} &= \hat{\mathbf{y}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}\times\hat{\mathbf{z}} = \mathbf{0},\\ \hat{\mathbf{x}}\times\hat{\mathbf{y}} &= -\hat{\mathbf{y}}\times\hat{\mathbf{x}} = \hat{\mathbf{z}},\\ \hat{\mathbf{y}}\times\hat{\mathbf{z}} &= -\hat{\mathbf{z}}\times\hat{\mathbf{y}} = \hat{\mathbf{x}},\\ \hat{\mathbf{z}}\times\hat{\mathbf{x}} &= -\hat{\mathbf{x}}\times\hat{\mathbf{z}} = \hat{\mathbf{y}}.\end{aligned} \tag{1.12}$$

[1]These signs pertain to a *right-handed* coordinate system (x-axis out of the page, y-axis to the right, z-axis up, or any rotated version thereof). In a *left-handed* system (z-axis down), the signs would be reversed: $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = -\hat{\mathbf{z}}$, and so on. We shall use right-handed systems exclusively.

Therefore,

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \times (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}) \\ &= (A_yB_z - A_zB_y)\hat{\mathbf{x}} + (A_zB_x - A_xB_z)\hat{\mathbf{y}} + (A_xB_y - A_yB_x)\hat{\mathbf{z}}.\end{aligned} \tag{1.13}$$

This cumbersome expression can be written more neatly as a determinant:

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \tag{1.14}$$

Rule (iv): *To calculate the cross product, form the determinant whose first row is* $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$, *whose second row is* **A** *(in component form), and whose third row is* **B**.

Example 1.2. Find the angle between the face diagonals of a cube.

Solution

We might as well use a cube of side 1, and place it as shown in Fig. 1.10, with one corner at the origin. The face diagonals **A** and **B** are

$$\mathbf{A} = 1\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 1\,\hat{\mathbf{z}}; \qquad \mathbf{B} = 0\,\hat{\mathbf{x}} + 1\,\hat{\mathbf{y}} + 1\,\hat{\mathbf{z}}.$$

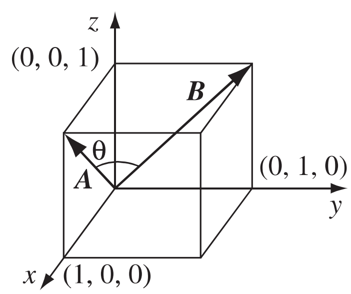


FIGURE 1.10

So, in component form,

$$\mathbf{A}\cdot\mathbf{B} = 1\cdot 0 + 0\cdot 1 + 1\cdot 1 = 1.$$

On the other hand, in "abstract" form,

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta = \sqrt{2}\sqrt{2}\cos\theta = 2\cos\theta.$$

Therefore,

$$\cos\theta = 1/2, \qquad \text{or} \qquad \theta = 60^\circ.$$

Of course, you can get the answer more easily by drawing in a diagonal across the top of the cube, completing the equilateral triangle. But in cases where the geometry is not so simple, this device of comparing the abstract and component forms of the dot product can be a very efficient means of finding angles.

Problem 1.3 Find the angle between the body diagonals of a cube.

Problem 1.4 Use the cross product to find the components of the unit vector $\hat{\mathbf{n}}$ perpendicular to the shaded plane in Fig. 1.11.

1.1.3 ■ Triple Products

Since the cross product of two vectors is itself a vector, it can be dotted or crossed with a third vector to form a *triple* product.

(i) Scalar triple product: $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$. Geometrically, $|\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})|$ is the volume of the parallelepiped generated by $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, since $|\mathbf{B} \times \mathbf{C}|$ is the area of the base, and $|\mathbf{A} \cos\theta|$ is the altitude (Fig. 1.12). Evidently,

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}), \tag{1.15}$$

for they all correspond to the same figure. Note that "alphabetical" order is preserved—in view of Eq. 1.6, the "nonalphabetical" triple products,

$$\mathbf{A} \cdot (\mathbf{C} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{A} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{B} \times \mathbf{A}),$$

have the opposite sign. In component form,

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}. \tag{1.16}$$

Note that the dot and cross can be interchanged:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$$

(this follows immediately from Eq. 1.15); however, the placement of the parentheses is critical: $(\mathbf{A} \cdot \mathbf{B}) \times \mathbf{C}$ is a meaningless expression—you can't make a cross product from a *scalar* and a vector.

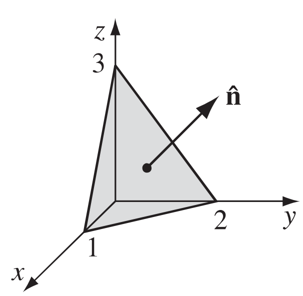


FIGURE 1.11

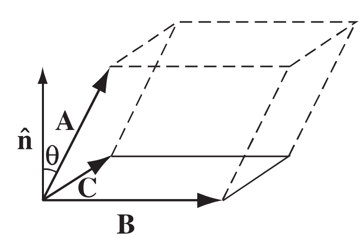


FIGURE 1.12

(ii) Vector triple product: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$. The vector triple product can be simplified by the so-called **BAC-CAB** rule:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}). \tag{1.17}$$

Notice that

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = -\mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = -\mathbf{A}(\mathbf{B} \cdot \mathbf{C}) + \mathbf{B}(\mathbf{A} \cdot \mathbf{C})$$

is an entirely different vector (cross-products are not associative). All *higher* vector products can be similarly reduced, often by repeated application of Eq. 1.17, so it is never necessary for an expression to contain more than one cross product in any term. For instance,

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C});$$

$$\mathbf{A} \times [\mathbf{B} \times (\mathbf{C} \times \mathbf{D})] = \mathbf{B}[\mathbf{A} \cdot (\mathbf{C} \times \mathbf{D})] - (\mathbf{A} \cdot \mathbf{B})(\mathbf{C} \times \mathbf{D}). \tag{1.18}$$

Problem 1.5 Prove the **BAC-CAB** rule by writing out both sides in component form.

Problem 1.6 Prove that

$$[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] + [\mathbf{B} \times (\mathbf{C} \times \mathbf{A})] + [\mathbf{C} \times (\mathbf{A} \times \mathbf{B})] = \mathbf{0}.$$

Under what conditions does $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$?

1.1.4 ■ Position, Displacement, and Separation Vectors

The location of a point in three dimensions can be described by listing its Cartesian coordinates (x, y, z). The vector to that point from the origin ($\mathcal{O}$) is called the **position vector** (Fig. 1.13):

$$\mathbf{r} \equiv x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}. \tag{1.19}$$

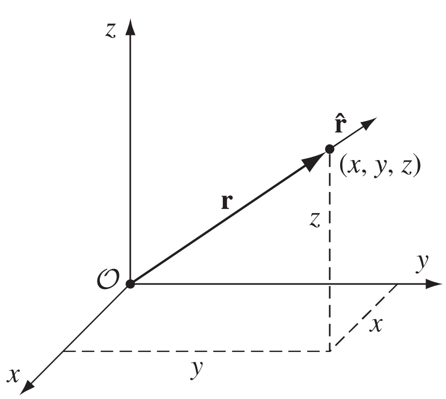


FIGURE 1.13

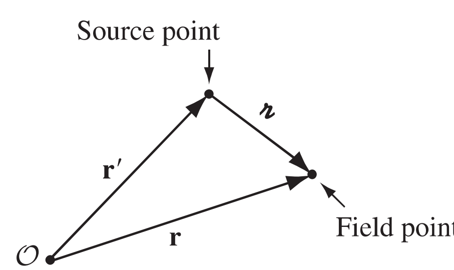


FIGURE 1.14

I will reserve the letter **r** for this purpose, throughout the book. Its magnitude,

$$r = \sqrt{x^2 + y^2 + z^2}, \tag{1.20}$$

is the distance from the origin, and

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}} \tag{1.21}$$

is a unit vector pointing radially outward. The **infinitesimal displacement vector**, from (x, y, z) to $(x + dx, y + dy, z + dz)$, is

$$d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}. \tag{1.22}$$

(We could call this $d\mathbf{r}$, since that's what it *is*, but it is useful to have a special notation for infinitesimal displacements.)

In electrodynamics, one frequently encounters problems involving *two* points—typically, a **source point**, $\mathbf{r}'$, where an electric charge is located, and a **field point**, $\mathbf{r}$, at which you are calculating the electric or magnetic field (Fig. 1.14). It pays to adopt right from the start some short-hand notation for the **separation vector** from the source point to the field point. I shall use for this purpose the script letter $\boldsymbol{\mathscr{r}}$:

$$\boldsymbol{\mathscr{r}} \equiv \mathbf{r} - \mathbf{r}'. \tag{1.23}$$

Its magnitude is

$$\mathscr{r} = |\mathbf{r} - \mathbf{r}'|, \tag{1.24}$$

and a unit vector in the direction from $\mathbf{r}'$ to $\mathbf{r}$ is

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}. \tag{1.25}$$

In Cartesian coordinates,

$$\boldsymbol{\mathscr{r}} = (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}, \tag{1.26}$$

$$\mathscr{r} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}, \tag{1.27}$$

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{(x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \tag{1.28}$$

(from which you can appreciate the economy of the script-$\mathscr{r}$ notation).

Problem 1.7 Find the separation vector $\boldsymbol{\mathscr{r}}$ from the source point (2,8,7) to the field point (4,6,8). Determine its magnitude ($\mathscr{r}$), and construct the unit vector $\hat{\boldsymbol{\mathscr{r}}}$.

1.1.5 ■ How Vectors Transform[2]

The definition of a vector as "a quantity with a magnitude and direction" is not altogether satisfactory: What precisely does "direction" *mean*? This may seem a pedantic question, but we shall soon encounter a species of derivative that *looks* rather like a vector, and we'll want to know for sure whether it *is* one.

You might be inclined to say that a vector is anything that has three components that combine properly under addition. Well, how about this: We have a barrel of fruit that contains N_x pears, N_y apples, and N_z bananas. Is $\mathbf{N} = N_x\hat{\mathbf{x}} + N_y\hat{\mathbf{y}} + N_z\hat{\mathbf{z}}$ a vector? It has three components, and when you add another barrel with M_x pears, M_y apples, and M_z bananas the result is $(N_x + M_x)$ pears, $(N_y + M_y)$ apples, $(N_z + M_z)$ bananas. So it does *add* like a vector. Yet it's obviously *not* a vector, in the physicist's sense of the word, because it doesn't really have a direction. What exactly is wrong with it?

The answer is that **N** *does not transform properly when you change coordinates.* The coordinate frame we use to describe positions in space is of course entirely arbitrary, but there is a specific geometrical transformation law for converting vector components from one frame to another. Suppose, for instance, the $\overline{x}, \overline{y}, \overline{z}$ system is rotated by angle ϕ, relative to x, y, z, about the common $x = \overline{x}$ axes. From Fig. 1.15,

$$A_y = A\cos\theta, \qquad A_z = A\sin\theta,$$

while

$$\begin{aligned}
\overline{A}_y &= A\cos\overline{\theta} = A\cos(\theta-\phi) = A(\cos\theta\cos\phi + \sin\theta\sin\phi)\\
&= \cos\phi A_y + \sin\phi A_z,\\
\overline{A}_z &= A\sin\overline{\theta} = A\sin(\theta-\phi) = A(\sin\theta\cos\phi - \cos\theta\sin\phi)\\
&= -\sin\phi A_y + \cos\phi A_z.
\end{aligned}$$

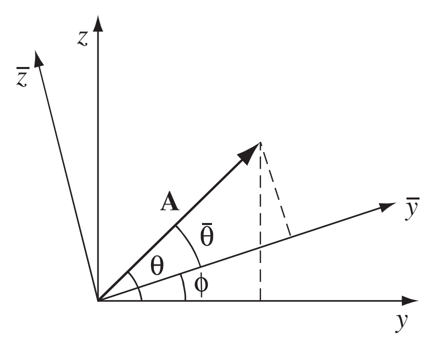


FIGURE 1.15

[2]This section can be skipped without loss of continuity.

We might express this conclusion in matrix notation:

$$\begin{pmatrix} \overline{A}_y \\ \overline{A}_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_y \\ A_z \end{pmatrix}. \tag{1.29}$$

More generally, for rotation about an *arbitrary* axis in three dimensions, the transformation law takes the form

$$\begin{pmatrix} \overline{A}_x \\ \overline{A}_y \\ \overline{A}_z \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}, \tag{1.30}$$

or, more compactly,

$$\overline{A}_i = \sum_{j=1}^{3} R_{ij} A_j, \tag{1.31}$$

where the index 1 stands for x, 2 for y, and 3 for z. The elements of the matrix R can be ascertained, for a given rotation, by the same sort of trigonometric arguments as we used for a rotation about the x axis.

Now: *Do* the components of $\mathbf{N}$ transform in this way? Of *course* not—it doesn't matter what coordinates you use to represent positions in space; there are still just as many apples in the barrel. You can't convert a pear into a banana by choosing a different set of axes, but you *can* turn A_x into $\overline{A}_y$. Formally, then, a *vector is any set of three components that transforms in the same manner as a displacement when you change coordinates.* As always, displacement is the *model* for the behavior of all vectors.[3]

By the way, a (second-rank) **tensor** is a quantity with *nine* components, T_{xx}, T_{xy}, T_{xz}, T_{yx}, …, T_{zz}, which transform with *two* factors of R:

$$\begin{aligned} \overline{T}_{xx} = & R_{xx}(R_{xx}T_{xx} + R_{xy}T_{xy} + R_{xz}T_{xz}) \\ & + R_{xy}(R_{xx}T_{yx} + R_{xy}T_{yy} + R_{xz}T_{yz}) \\ & + R_{xz}(R_{xx}T_{zx} + R_{xy}T_{zy} + R_{xz}T_{zz}), \dots \end{aligned}$$

or, more compactly,

$$\overline{T}_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} R_{ik} R_{jl} T_{kl}. \tag{1.32}$$

[3]If you're a mathematician you might want to contemplate generalized vector spaces in which the "axes" have nothing to do with direction and the basis vectors are no longer $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ (indeed, there may be more than three dimensions). This is the subject of **linear algebra**. But for our purposes all vectors live in ordinary 3-space (or, in Chapter 12, in 4-dimensional space-time.)

In general, an nth-rank tensor has n indices and 3^n components, and transforms with n factors of R. In this hierarchy, a vector is a tensor of rank 1, and a scalar is a tensor of rank zero.[4]

Problem 1.8

(a) Prove that the two-dimensional rotation matrix (Eq. 1.29) preserves dot products. (That is, show that $\overline{A}_y\overline{B}_y + \overline{A}_z\overline{B}_z = A_yB_y + A_zB_z$.)

(b) What constraints must the elements (R_{ij}) of the three-dimensional rotation matrix (Eq. 1.30) satisfy, in order to preserve the length of **A** (for all vectors **A**)?

Problem 1.9 Find the transformation matrix R that describes a rotation by 120° about an axis from the origin through the point (1, 1, 1). The rotation is clockwise as you look down the axis toward the origin.

Problem 1.10

(a) How do the components of a vector[5] transform under a **translation** of coordinates ($\overline{x} = x$, $\overline{y} = y - a$, $\overline{z} = z$, Fig. 1.16a)?

(b) How do the components of a vector transform under an **inversion** of coordinates ($\overline{x} = -x$, $\overline{y} = -y$, $\overline{z} = -z$, Fig. 1.16b)?

(c) How do the components of a cross product (Eq. 1.13) transform under inversion? [The cross-product of two vectors is properly called a **pseudovector** because of this "anomalous" behavior.] Is the cross product of two pseudovectors a vector, or a pseudovector? Name two pseudovector quantities in classical mechanics.

(d) How does the scalar triple product of three vectors transform under inversions? (Such an object is called a **pseudoscalar.**)

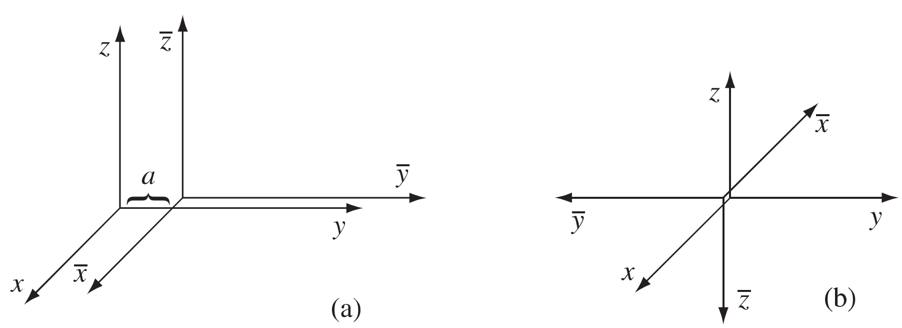


FIGURE 1.16

[4] A scalar does not change when you change coordinates. In particular, the components of a vector are *not* scalars, but the magnitude is.

[5] *Beware:* The vector **r** (Eq. 1.19) goes from a specific point in space (the origin, $\mathcal{O}$) to the point $P = (x, y, z)$. Under translations the *new* origin ($\bar{\mathcal{O}}$) is at a different location, and the arrow from $\bar{\mathcal{O}}$ to P is a completely different vector. The original vector **r** still goes from $\mathcal{O}$ to P, regardless of the coordinates used to label these points.

1.2 ■ DIFFERENTIAL CALCULUS

1.2.1 ■ "Ordinary" Derivatives

Suppose we have a function of one variable: $f(x)$. *Question:* What does the derivative, df/dx, do for us? *Answer:* It tells us how rapidly the function $f(x)$ varies when we change the argument x by a tiny amount, dx:

$$df = \left(\frac{df}{dx}\right) dx. \tag{1.33}$$

In words: If we increment x by an infinitesimal amount dx, then f changes by an amount df; the derivative is the proportionality factor. For example, in Fig. 1.17(a), the function varies slowly with x, and the derivative is correspondingly small. In Fig. 1.17(b), f increases rapidly with x, and the derivative is large, as you move away from $x = 0$.

Geometrical Interpretation: The derivative df/dx is the *slope* of the graph of f versus x.

1.2.2 ■ Gradient

Suppose, now, that we have a function of *three* variables—say, the temperature $T(x, y, z)$ in this room. (Start out in one corner, and set up a system of axes; then for each point (x, y, z) in the room, T gives the temperature at that spot.) We want to generalize the notion of "derivative" to functions like T, which depend not on *one* but on *three* variables.

A derivative is supposed to tell us how fast the function varies, if we move a little distance. But this time the situation is more complicated, because it depends on what *direction* we move: If we go straight up, then the temperature will probably increase fairly rapidly, but if we move horizontally, it may not change much at all. In fact, the question "How fast does T vary?" has an infinite number of answers, one for each direction we might choose to explore.

Fortunately, the problem is not as bad as it looks. A theorem on partial derivatives states that

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz. \tag{1.34}$$

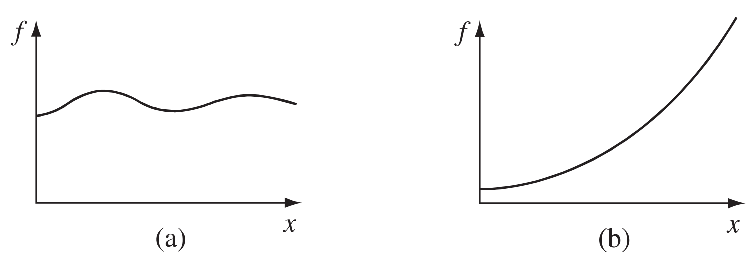


FIGURE 1.17

This tells us how T changes when we alter all three variables by the infinitesimal amounts dx, dy, dz. Notice that we do *not* require an infinite number of derivatives—*three* will suffice: the *partial* derivatives along each of the three coordinate directions.

Equation 1.34 is reminiscent of a dot product:

$$\begin{aligned} dT &= \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right) \cdot (dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}) \\ &= (\nabla T) \cdot (d\mathbf{l}), \end{aligned} \tag{1.35}$$

where

$$\nabla T \equiv \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{1.36}$$

is the **gradient** of T. Note that ∇T is a *vector* quantity, with three components; it is the generalized derivative we have been looking for. Equation 1.35 is the three-dimensional version of Eq. 1.33.

Geometrical Interpretation of the Gradient: Like any vector, the gradient has *magnitude* and *direction*. To determine its geometrical meaning, let's rewrite the dot product (Eq. 1.35) using Eq. 1.1:

$$dT = \nabla T \cdot d\mathbf{l} = |\nabla T||d\mathbf{l}|\cos\theta, \tag{1.37}$$

where θ is the angle between ∇T and $d\mathbf{l}$. Now, if we *fix* the *magnitude* $|d\mathbf{l}|$ and search around in various *directions* (that is, vary θ), the *maximum* change in T evidentally occurs when $\theta = 0$ (for then $\cos\theta = 1$). That is, for a fixed distance $|d\mathbf{l}|$, dT is greatest when I move in the *same direction* as ∇T. Thus:

> *The gradient ∇T points in the direction of maximum increase of the function T.*

Moreover:

> *The magnitude $|\nabla T|$ gives the slope (rate of increase) along this maximal direction.*

Imagine you are standing on a hillside. Look all around you, and find the direction of steepest ascent. That is the *direction* of the gradient. Now measure the *slope* in that direction (rise over run). That is the *magnitude* of the gradient. (Here the function we're talking about is the height of the hill, and the coordinates it depends on are positions—latitude and longitude, say. This function depends on only *two* variables, not *three*, but the geometrical meaning of the gradient is easier to grasp in two dimensions.) Notice from Eq. 1.37 that the direction of maximum *descent* is opposite to the direction of maximum *ascent*, while at right angles ($\theta = 90°$) the slope is zero (the gradient is perpendicular to the contour lines). You can conceive of surfaces that do not have these properties, but they always have "kinks" in them, and correspond to nondifferentiable functions.

What would it mean for the gradient to vanish? If $\nabla T = \mathbf{0}$ at (x, y, z), then $dT = 0$ for small displacements about the point (x, y, z). This is, then, a **stationary point** of the function $T(x, y, z)$. It could be a maximum (a summit), a minimum (a valley), a saddle point (a pass), or a "shoulder." This is analogous to the situation for functions of *one* variable, where a vanishing derivative signals a maximum, a minimum, or an inflection. In particular, if you want to locate the extrema of a function of three variables, set its gradient equal to zero.

Example 1.3. Find the gradient of $r = \sqrt{x^2 + y^2 + z^2}$ (the magnitude of the position vector).

Solution

$$\begin{aligned}\nabla r &= \frac{\partial r}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial r}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial r}{\partial z}\,\hat{\mathbf{z}} \\ &= \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}}\,\hat{\mathbf{x}} + \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2+z^2}}\,\hat{\mathbf{y}} + \frac{1}{2}\frac{2z}{\sqrt{x^2+y^2+z^2}}\,\hat{\mathbf{z}} \\ &= \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{\sqrt{x^2+y^2+z^2}} = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}.\end{aligned}$$

Does this make sense? Well, it says that the distance from the origin increases most rapidly in the radial direction, and that its *rate* of increase in that direction is 1. . . just what you'd expect.

Problem 1.11 Find the gradients of the following functions:

(a) $f(x, y, z) = x^2 + y^3 + z^4$.

(b) $f(x, y, z) = x^2 y^3 z^4$.

(c) $f(x, y, z) = e^x \sin(y) \ln(z)$.

Problem 1.12 The height of a certain hill (in feet) is given by

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12),$$

where y is the distance (in miles) north, x the distance east of South Hadley.

(a) Where is the top of the hill located?

(b) How high is the hill?

(c) How steep is the slope (in feet per mile) at a point 1 mile north and one mile east of South Hadley? In what direction is the slope steepest, at that point?

• **Problem 1.13** Let $\boldsymbol{\mathscr{r}}$ be the separation vector from a fixed point (x', y', z') to the point (x, y, z), and let $\mathscr{r}$ be its length. Show that

(a) $\nabla(\mathscr{r}^2) = 2\boldsymbol{\mathscr{r}}$.

(b) $\nabla(1/\mathscr{r}) = -\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$.

(c) What is the *general* formula for $\nabla(\mathscr{r}^n)$?

! **Problem 1.14** Suppose that f is a function of two variables (y and z) only. Show that the gradient $\nabla f = (\partial f/\partial y)\hat{\mathbf{y}} + (\partial f/\partial z)\hat{\mathbf{z}}$ transforms as a vector under rotations, Eq. 1.29. [*Hint:* $(\partial f/\partial \overline{y}) = (\partial f/\partial y)(\partial y/\partial \overline{y}) + (\partial f/\partial z)(\partial z/\partial \overline{y})$, and the analogous formula for $\partial f/\partial \overline{z}$. We know that $\overline{y} = y\cos\phi + z\sin\phi$ and $\overline{z} = -y\sin\phi + z\cos\phi$; "solve" these equations for y and z (as functions of $\overline{y}$ and $\overline{z}$), and compute the needed derivatives $\partial y/\partial \overline{y}$, $\partial z/\partial \overline{y}$, etc.]

1.2.3 ■ The Del Operator

The gradient has the formal appearance of a vector, ∇, "multiplying" a scalar T:

$$\nabla T = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)T. \tag{1.38}$$

(For once, I write the unit vectors to the *left*, just so no one will think this means $\partial\hat{\mathbf{x}}/\partial x$, and so on—which would be zero, since $\hat{\mathbf{x}}$ is constant.) The term in parentheses is called **del**:

$$\boxed{\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}.} \tag{1.39}$$

Of course, del is *not* a vector, in the usual sense. Indeed, it doesn't mean much until we provide it with a function to act upon. Furthermore, it does not "multiply" T; rather, it is an instruction to *differentiate* what follows. To be precise, then, we say that ∇ is a **vector operator** that *acts upon* T, not a vector that multiplies T.

With this qualification, though, ∇ mimics the behavior of an ordinary vector in virtually every way; almost anything that can be done with other vectors can also be done with ∇, if we merely translate "multiply" by "act upon." So by all means take the vector appearance of ∇ seriously: it is a marvelous piece of notational simplification, as you will appreciate if you ever consult Maxwell's original work on electromagnetism, written without the benefit of ∇.

Now, an ordinary vector $\mathbf{A}$ can multiply in three ways:

1. By a scalar a : $\mathbf{A}a$;
2. By a vector $\mathbf{B}$, via the dot product: $\mathbf{A}\cdot\mathbf{B}$;
3. By a vector $\mathbf{B}$ via the cross product: $\mathbf{A}\times\mathbf{B}$.

Correspondingly, there are three ways the operator ∇ can act:

1. On a scalar function T : ∇T (the gradient);
2. On a vector function $\mathbf{v}$, via the dot product: $\nabla\cdot\mathbf{v}$ (the **divergence**);
3. On a vector function $\mathbf{v}$, via the cross product: $\nabla\times\mathbf{v}$ (the **curl**).

We have already discussed the gradient. In the following sections we examine the other two vector derivatives: divergence and curl.

1.2.4 ■ The Divergence

From the definition of ∇ we construct the divergence:

$$\nabla \cdot \mathbf{v} = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot (v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}})$$
$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}. \qquad (1.40)$$

Observe that the divergence of a vector function[6] $\mathbf{v}$ is itself a *scalar* $\nabla \cdot \mathbf{v}$.

Geometrical Interpretation: The name **divergence** is well chosen, for $\nabla \cdot \mathbf{v}$ is a measure of how much the vector $\mathbf{v}$ spreads out (diverges) from the point in question. For example, the vector function in Fig. 1.18a has a large (positive) divergence (if the arrows pointed *in*, it would be a *negative* divergence), the function in Fig. 1.18b has zero divergence, and the function in Fig. 1.18c again has a positive divergence. (Please understand that $\mathbf{v}$ here is a *function*—there's a different vector associated with every point in space. In the diagrams, of course, I can only draw the arrows at a few representative locations.)

Imagine standing at the edge of a pond. Sprinkle some sawdust or pine needles on the surface. If the material spreads out, then you dropped it at a point of positive divergence; if it collects together, you dropped it at a point of negative divergence. (The vector function $\mathbf{v}$ in this model is the velocity of the water at the surface—this is a *two*-dimensional example, but it helps give one a "feel" for what the divergence means. A point of positive divergence is a source, or "faucet"; a point of negative divergence is a sink, or "drain.")

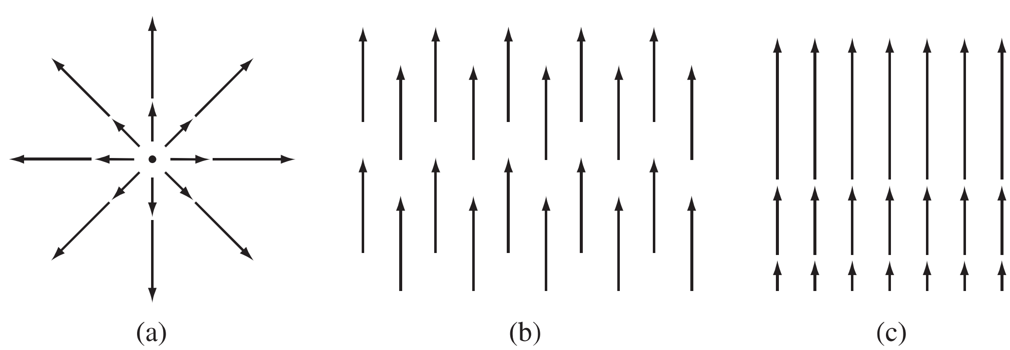


FIGURE 1.18

[6]A vector function $\mathbf{v}(x, y, z) = v_x(x, y, z)\,\hat{\mathbf{x}} + v_y(x, y, z)\,\hat{\mathbf{y}} + v_z(x, y, z)\,\hat{\mathbf{z}}$ is really *three* functions—one for each component. There's no such thing as the divergence of a scalar.

Example 1.4. Suppose the functions in Fig. 1.18 are $\mathbf{v}_a = \mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$, $\mathbf{v}_b = \hat{\mathbf{z}}$, and $\mathbf{v}_c = z\,\hat{\mathbf{z}}$. Calculate their divergences.

Solution

$$\nabla \cdot \mathbf{v}_a = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3.$$

As anticipated, this function has a positive divergence.

$$\nabla \cdot \mathbf{v}_b = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(1) = 0 + 0 + 0 = 0,$$

as expected.

$$\nabla \cdot \mathbf{v}_c = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(z) = 0 + 0 + 1 = 1.$$

Problem 1.15 Calculate the divergence of the following vector functions:

(a) $\mathbf{v}_a = x^2\,\hat{\mathbf{x}} + 3xz^2\,\hat{\mathbf{y}} - 2xz\,\hat{\mathbf{z}}$.

(b) $\mathbf{v}_b = xy\,\hat{\mathbf{x}} + 2yz\,\hat{\mathbf{y}} + 3zx\,\hat{\mathbf{z}}$.

(c) $\mathbf{v}_c = y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + 2yz\,\hat{\mathbf{z}}$.

• **Problem 1.16** Sketch the vector function

$$\mathbf{v} = \frac{\hat{\mathbf{r}}}{r^2},$$

and compute its divergence. The answer may surprise you. . . can you explain it?

! **Problem 1.17** In two dimensions, show that the divergence transforms as a scalar under rotations. [*Hint:* Use Eq. 1.29 to determine $\bar{v}_y$ and $\bar{v}_z$, and the method of Prob. 1.14 to calculate the derivatives. Your aim is to show that $\partial\bar{v}_y/\partial\bar{y} + \partial\bar{v}_z/\partial\bar{z} = \partial v_y/\partial y + \partial v_z/\partial z$.]

1.2.5 ■ The Curl

From the definition of ∇ we construct the curl:

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right). \quad (1.41)$$

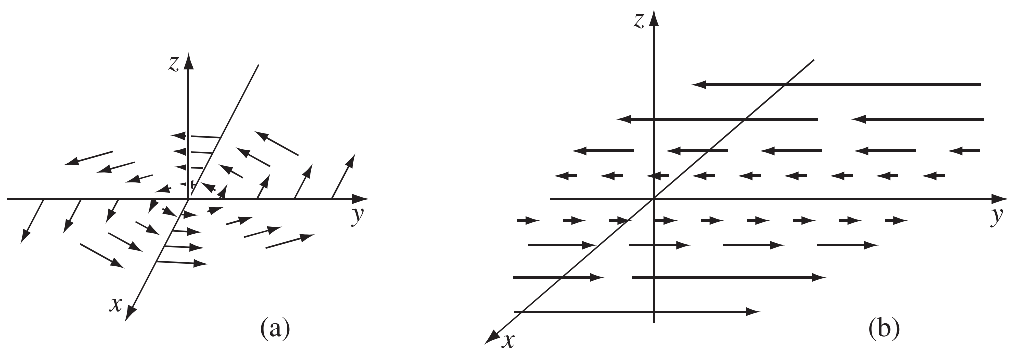


FIGURE 1.19

Notice that the curl of a vector function[7] **v** is, like any cross product, a *vector*.

Geometrical Interpretation: The name **curl** is also well chosen, for $\nabla \times \mathbf{v}$ is a measure of how much the vector **v** swirls around the point in question. Thus the three functions in Fig. 1.18 all have zero curl (as you can easily check for yourself), whereas the functions in Fig. 1.19 have a substantial curl, pointing in the z direction, as the natural right-hand rule would suggest. Imagine (again) you are standing at the edge of a pond. Float a small paddlewheel (a cork with toothpicks pointing out radially would do); if it starts to rotate, then you placed it at a point of nonzero *curl.* A whirlpool would be a region of large curl.

Example 1.5. Suppose the function sketched in Fig. 1.19a is $\mathbf{v}_a = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$, and that in Fig. 1.19b is $\mathbf{v}_b = x\hat{\mathbf{y}}$. Calculate their curls.

Solution

$$\nabla \times \mathbf{v}_a = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = 2\hat{\mathbf{z}},$$

and

$$\nabla \times \mathbf{v}_b = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & x & 0 \end{vmatrix} = \hat{\mathbf{z}}.$$

As expected, these curls point in the $+z$ direction. (Incidentally, they both have zero divergence, as you might guess from the pictures: nothing is "spreading out"... it just "swirls around.")

---

[7] There's no such thing as the curl of a scalar.

Problem 1.18 Calculate the curls of the vector functions in Prob. 1.15.

Problem 1.19 Draw a circle in the xy plane. At a few representative points draw the vector $\mathbf{v}$ tangent to the circle, pointing in the clockwise direction. By comparing adjacent vectors, determine the *sign* of $\partial v_x/\partial y$ and $\partial v_y/\partial x$. According to Eq. 1.41, then, what is the direction of $\nabla \times \mathbf{v}$? Explain how this example illustrates the geometrical interpretation of the curl.

Problem 1.20 Construct a vector function that has zero divergence and zero curl everywhere. (A *constant* will do the job, of course, but make it something a little more interesting than that!)

1.2.6 ■ Product Rules

The calculation of ordinary derivatives is facilitated by a number of rules, such as the sum rule:

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx},$$

the rule for multiplying by a constant:

$$\frac{d}{dx}(kf) = k\frac{df}{dx},$$

the product rule:

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx},$$

and the quotient rule:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}.$$

Similar relations hold for the vector derivatives. Thus,

$$\boldsymbol{\nabla}(f+g) = \boldsymbol{\nabla} f + \boldsymbol{\nabla} g, \qquad \boldsymbol{\nabla}\cdot(\mathbf{A}+\mathbf{B}) = (\boldsymbol{\nabla}\cdot\mathbf{A}) + (\boldsymbol{\nabla}\cdot\mathbf{B}),$$

$$\boldsymbol{\nabla}\times(\mathbf{A}+\mathbf{B}) = (\boldsymbol{\nabla}\times\mathbf{A}) + (\boldsymbol{\nabla}\times\mathbf{B}),$$

and

$$\boldsymbol{\nabla}(kf) = k\boldsymbol{\nabla} f, \qquad \boldsymbol{\nabla}\cdot(k\mathbf{A}) = k(\boldsymbol{\nabla}\cdot\mathbf{A}), \qquad \boldsymbol{\nabla}\times(k\mathbf{A}) = k(\boldsymbol{\nabla}\times\mathbf{A}),$$

as you can check for yourself. The product rules are not quite so simple. There are two ways to construct a scalar as the product of two functions:

fg (product of two scalar functions),

$\mathbf{A}\cdot\mathbf{B}$ (dot product of two vector functions), and two ways to make a vector:

$$\begin{aligned} f\mathbf{A} \quad & \text{(scalar times vector)}, \\ \mathbf{A}\times\mathbf{B} \quad & \text{(cross product of two vectors)}. \end{aligned}$$

Accordingly, there are *six* product rules, two for gradients:

(i) $$\nabla(fg) = f\nabla g + g\nabla f,$$

(ii) $$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A},$$

two for divergences:

(iii) $$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f),$$

(iv) $$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}),$$

and two for curls:

(v) $$\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f),$$

(vi) $$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}).$$

You will be using these product rules so frequently that I have put them inside the front cover for easy reference. The proofs come straight from the product rule for ordinary derivatives. For instance,

$$\begin{aligned} \nabla\cdot(f\mathbf{A}) &= \frac{\partial}{\partial x}(fA_x) + \frac{\partial}{\partial y}(fA_y) + \frac{\partial}{\partial z}(fA_z) \\ &= \left(\frac{\partial f}{\partial x}A_x + f\frac{\partial A_x}{\partial x}\right) + \left(\frac{\partial f}{\partial y}A_y + f\frac{\partial A_y}{\partial y}\right) + \left(\frac{\partial f}{\partial z}A_z + f\frac{\partial A_z}{\partial z}\right) \\ &= (\nabla f)\cdot\mathbf{A} + f(\nabla\cdot\mathbf{A}). \end{aligned}$$

It is also possible to formulate three quotient rules:

$$\begin{aligned} \nabla\left(\frac{f}{g}\right) &= \frac{g\nabla f - f\nabla g}{g^2}, \\ \nabla\cdot\left(\frac{\mathbf{A}}{g}\right) &= \frac{g(\nabla\cdot\mathbf{A}) - \mathbf{A}\cdot(\nabla g)}{g^2}, \\ \nabla\times\left(\frac{\mathbf{A}}{g}\right) &= \frac{g(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla g)}{g^2}. \end{aligned}$$

However, since these can be obtained quickly from the corresponding product rules, there is no point in listing them separately.

Problem 1.21 Prove product rules (i), (iv), and (v).

Problem 1.22

(a) If $\mathbf{A}$ and $\mathbf{B}$ are two vector functions, what does the expression $(\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B}$ mean? (That is, what are its x, y, and z components, in terms of the Cartesian components of $\mathbf{A}$, $\mathbf{B}$, and $\boldsymbol{\nabla}$?)

(b) Compute $(\hat{\mathbf{r}} \cdot \boldsymbol{\nabla})\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is the unit vector defined in Eq. 1.21.

(c) For the functions in Prob. 1.15, evaluate $(\mathbf{v}_a \cdot \boldsymbol{\nabla})\mathbf{v}_b$.

Problem 1.23 (For masochists only.) Prove product rules (ii) and (vi). Refer to Prob. 1.22 for the definition of $(\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B}$.

Problem 1.24 Derive the three quotient rules.

Problem 1.25

(a) Check product rule (iv) (by calculating each term separately) for the functions

$$\mathbf{A} = x\,\hat{\mathbf{x}} + 2y\,\hat{\mathbf{y}} + 3z\,\hat{\mathbf{z}}; \qquad \mathbf{B} = 3y\,\hat{\mathbf{x}} - 2x\,\hat{\mathbf{y}}.$$

(b) Do the same for product rule (ii).

(c) Do the same for rule (vi).

1.2.7 ■ Second Derivatives

The gradient, the divergence, and the curl are the only first derivatives we can make with $\boldsymbol{\nabla}$; by applying $\boldsymbol{\nabla}$ *twice*, we can construct five species of *second* derivatives. The gradient $\boldsymbol{\nabla}T$ is a *vector*, so we can take the *divergence* and *curl* of it:

(1) Divergence of gradient: $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}T)$.

(2) Curl of gradient: $\boldsymbol{\nabla} \times (\boldsymbol{\nabla}T)$.

The divergence $\boldsymbol{\nabla} \cdot \mathbf{v}$ is a *scalar*—all we can do is take its *gradient:*

(3) Gradient of divergence: $\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{v})$.

The curl $\boldsymbol{\nabla} \times \mathbf{v}$ is a *vector*, so we can take its *divergence* and *curl:*

(4) Divergence of curl: $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{v})$.

(5) Curl of curl: $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v})$.

This exhausts the possibilities, and in fact not all of them give anything new. Let's consider them one at a time:

$$(1) \qquad \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}T) = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right)$$
$$= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}. \qquad (1.42)$$

This object, which we write as $\nabla^2 T$ for short, is called the **Laplacian** of T; we shall be studying it in great detail later on. Notice that the Laplacian of a *scalar* T is a *scalar*. Occasionally, we shall speak of the Laplacian of a *vector,* $\nabla^2\mathbf{v}$. By this we mean a *vector* quantity whose x-component is the Laplacian of v_x, and so on:[8]

$$\nabla^2\mathbf{v} \equiv (\nabla^2 v_x)\hat{\mathbf{x}} + (\nabla^2 v_y)\hat{\mathbf{y}} + (\nabla^2 v_z)\hat{\mathbf{z}}. \tag{1.43}$$

This is nothing more than a convenient extension of the meaning of ∇^2.

(2) The curl of a gradient is always zero:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} T) = \mathbf{0}. \tag{1.44}$$

This is an important fact, which we shall use repeatedly; you can easily prove it from the definition of $\boldsymbol{\nabla}$, Eq. 1.39. *Beware*: You might think Eq. 1.44 is "obviously" true—isn't it just $(\boldsymbol{\nabla} \times \boldsymbol{\nabla})T$, and isn't the cross product of *any* vector (in this case, $\boldsymbol{\nabla}$) with itself always zero? This reasoning is suggestive, but not quite conclusive, since $\boldsymbol{\nabla}$ is an *operator* and does not "multiply" in the usual way. The proof of Eq. 1.44, in fact, hinges on the equality of cross derivatives:

$$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial x}\right). \tag{1.45}$$

If you think I'm being fussy, test your intuition on this one:

$$(\boldsymbol{\nabla} T) \times (\boldsymbol{\nabla} S).$$

Is *that* always zero? (It *would* be, of course, if you replaced the $\boldsymbol{\nabla}$'s by an ordinary vector.)

(3) $\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{v})$ seldom occurs in physical applications, and it has not been given any special name of its own—it's just **the gradient of the divergence.** Notice that $\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{v})$ is *not* the same as the Laplacian of a vector: $\nabla^2\mathbf{v} = (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\mathbf{v} \neq \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{v})$.

(4) The divergence of a curl, like the curl of a gradient, is always zero:

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{v}) = 0. \tag{1.46}$$

You can prove this for yourself. (Again, there is a fraudulent short-cut proof, using the vector identity $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$.)

(5) As you can check from the definition of $\boldsymbol{\nabla}$:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{v}) - \nabla^2\mathbf{v}. \tag{1.47}$$

So curl-of-curl gives nothing new; the first term is just number (3), and the second is the Laplacian (of a vector). (In fact, Eq. 1.47 is often used to *define* the

[8]In curvilinear coordinates, where the unit vectors themselves depend on position, they too must be differentiated (see Sect. 1.4.1).

Laplacian of a vector, in preference to Eq. 1.43, which makes explicit reference to Cartesian coordinates.)

Really, then, there are just two kinds of second derivatives: the Laplacian (which is of fundamental importance) and the gradient-of-divergence (which we seldom encounter). We could go through a similar ritual to work out *third* derivatives, but fortunately second derivatives suffice for practically all physical applications.

A final word on vector differential calculus: It *all* flows from the operator ∇, and from taking seriously its vectorial character. Even if you remembered *only* the definition of ∇, you could easily reconstruct all the rest.

Problem 1.26 Calculate the Laplacian of the following functions:

(a) $T_a = x^2 + 2xy + 3z + 4$.

(b) $T_b = \sin x \sin y \sin z$.

(c) $T_c = e^{-5x} \sin 4y \cos 3z$.

(d) $\mathbf{v} = x^2\,\hat{\mathbf{x}} + 3xz^2\,\hat{\mathbf{y}} - 2xz\,\hat{\mathbf{z}}$.

Problem 1.27 Prove that the divergence of a curl is always zero. *Check* it for function $\mathbf{v}_a$ in Prob. 1.15.

Problem 1.28 Prove that the curl of a gradient is always zero. *Check* it for function (b) in Prob. 1.11.

1.3 ■ INTEGRAL CALCULUS

1.3.1 ■ Line, Surface, and Volume Integrals

In electrodynamics, we encounter several different kinds of integrals, among which the most important are **line** (or **path**) **integrals**, **surface integrals** (or **flux**), and **volume integrals**.

(a) **Line Integrals.** A line integral is an expression of the form

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{v} \cdot d\mathbf{l}, \tag{1.48}$$

where $\mathbf{v}$ is a vector function, $d\mathbf{l}$ is the infinitesimal displacement vector (Eq. 1.22), and the integral is to be carried out along a prescribed path $\mathcal{P}$ from point $\mathbf{a}$ to point $\mathbf{b}$ (Fig. 1.20). If the path in question forms a closed loop (that is, if $\mathbf{b} = \mathbf{a}$), I shall put a circle on the integral sign:

$$\oint \mathbf{v} \cdot d\mathbf{l}. \tag{1.49}$$

At each point on the path, we take the dot product of $\mathbf{v}$ (evaluated at that point) with the displacement $d\mathbf{l}$ to the next point on the path. To a physicist, the most familiar example of a line integral is the work done by a force $\mathbf{F}$: $W = \int \mathbf{F} \cdot d\mathbf{l}$.

Ordinarily, the value of a line integral depends critically on the path taken from $\mathbf{a}$ to $\mathbf{b}$, but there is an important special class of vector functions for which the line

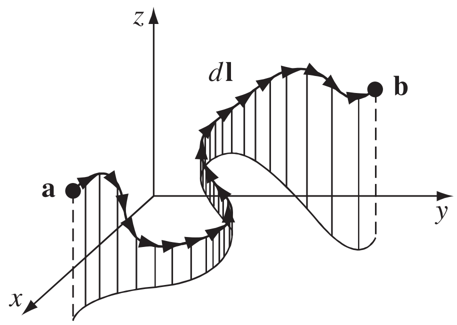


FIGURE 1.20

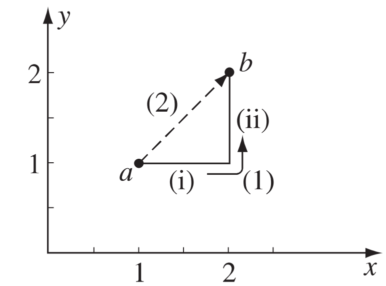


FIGURE 1.21

integral is *independent* of path and is determined entirely by the end points. It will be our business in due course to characterize this special class of vectors. (A *force* that has this property is called **conservative**.)

Example 1.6. Calculate the line integral of the function $\mathbf{v} = y^2\,\hat{\mathbf{x}} + 2x(y+1)\,\hat{\mathbf{y}}$ from the point $\mathbf{a} = (1, 1, 0)$ to the point $\mathbf{b} = (2, 2, 0)$, along the paths (1) and (2) in Fig. 1.21. What is $\oint \mathbf{v}\cdot d\mathbf{l}$ for the loop that goes from **a** to **b** along (1) and returns to **a** along (2)?

Solution

As always, $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$. Path (1) consists of two parts. Along the "horizontal" segment, $dy = dz = 0$, so

(i) $d\mathbf{l} = dx\,\hat{\mathbf{x}},\ y = 1,\ \mathbf{v}\cdot d\mathbf{l} = y^2\,dx = dx$, so $\int \mathbf{v}\cdot d\mathbf{l} = \int_1^2 dx = 1$.

On the "vertical" stretch, $dx = dz = 0$, so

(ii) $d\mathbf{l} = dy\,\hat{\mathbf{y}},\ x = 2,\ \mathbf{v}\cdot d\mathbf{l} = 2x(y+1)\,dy = 4(y+1)\,dy$, so

$$\int \mathbf{v}\cdot d\mathbf{l} = 4\int_1^2 (y+1)\,dy = 10.$$

By path (1), then,

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{v}\cdot d\mathbf{l} = 1 + 10 = 11.$$

Meanwhile, on path (2) $x = y,\ dx = dy$, and $dz = 0$, so
$d\mathbf{l} = dx\,\hat{\mathbf{x}} + dx\,\hat{\mathbf{y}},\ \mathbf{v}\cdot d\mathbf{l} = x^2\,dx + 2x(x+1)\,dx = (3x^2 + 2x)\,dx$,
and

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{v}\cdot d\mathbf{l} = \int_1^2 (3x^2 + 2x)\,dx = (x^3 + x^2)\Big|_1^2 = 10.$$

(The strategy here is to get everything in terms of one variable; I could just as well have eliminated x in favor of y.)

For the loop that goes *out* (1) and *back* (2), then,

$$\oint \mathbf{v} \cdot d\mathbf{l} = 11 - 10 = 1.$$

(b) **Surface Integrals.** A surface integral is an expression of the form

$$\int_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a}, \tag{1.50}$$

where $\mathbf{v}$ is again some vector function, and the integral is over a specified surface $\mathcal{S}$. Here $d\mathbf{a}$ is an infinitesimal patch of area, with direction perpendicular to the surface (Fig. 1.22). There are, of course, *two* directions perpendicular to any surface, so the *sign* of a surface integral is intrinsically ambiguous. If the surface is *closed* (forming a "balloon"), in which case I shall again put a circle on the integral sign

$$\oint \mathbf{v} \cdot d\mathbf{a},$$

then tradition dictates that "outward" is positive, but for open surfaces it's arbitrary. If $\mathbf{v}$ describes the flow of a fluid (mass per unit area per unit time), then $\int \mathbf{v} \cdot d\mathbf{a}$ represents the total mass per unit time passing through the surface—hence the alternative name, "flux."

Ordinarily, the value of a surface integral depends on the particular surface chosen, but there is a special class of vector functions for which it is *independent* of the surface and is determined entirely by the boundary line. An important task will be to characterize this special class of functions.

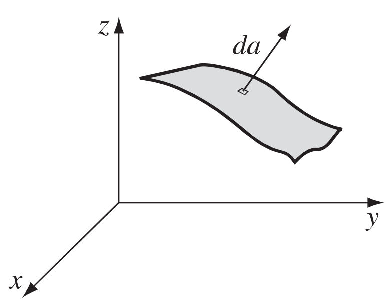


FIGURE 1.22

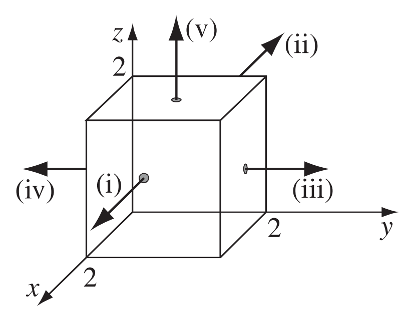


FIGURE 1.23

Example 1.7. Calculate the surface integral of $\mathbf{v} = 2xz\,\hat{\mathbf{x}} + (x+2)\,\hat{\mathbf{y}} + y(z^2-3)\,\hat{\mathbf{z}}$ over five sides (excluding the bottom) of the cubical box (side 2) in Fig. 1.23. Let "upward and outward" be the positive direction, as indicated by the arrows.

Solution

Taking the sides one at a time:

(i) $x = 2$, $d\mathbf{a} = dy\,dz\,\hat{\mathbf{x}}$, $\mathbf{v} \cdot d\mathbf{a} = 2xz\,dy\,dz = 4z\,dy\,dz$, so

$$\int \mathbf{v} \cdot d\mathbf{a} = 4 \int_0^2 dy \int_0^2 z\,dz = 16.$$

(ii) $x = 0$, $d\mathbf{a} = -dy\,dz\,\hat{\mathbf{x}}$, $\mathbf{v} \cdot d\mathbf{a} = -2xz\,dy\,dz = 0$, so

$$\int \mathbf{v} \cdot d\mathbf{a} = 0.$$

(iii) $y = 2$, $d\mathbf{a} = dx\,dz\,\hat{\mathbf{y}}$, $\mathbf{v} \cdot d\mathbf{a} = (x + 2)\,dx\,dz$, so

$$\int \mathbf{v} \cdot d\mathbf{a} = \int_0^2 (x + 2)\,dx \int_0^2 dz = 12.$$

(iv) $y = 0$, $d\mathbf{a} = -dx\,dz\,\hat{\mathbf{y}}$, $\mathbf{v} \cdot d\mathbf{a} = -(x + 2)\,dx\,dz$, so

$$\int \mathbf{v} \cdot d\mathbf{a} = -\int_0^2 (x + 2)\,dx \int_0^2 dz = -12.$$

(v) $z = 2$, $d\mathbf{a} = dx\,dy\,\hat{\mathbf{z}}$, $\mathbf{v} \cdot d\mathbf{a} = y(z^2 - 3)\,dx\,dy = y\,dx\,dy$, so

$$\int \mathbf{v} \cdot d\mathbf{a} = \int_0^2 dx \int_0^2 y\,dy = 4.$$

The *total* flux is

$$\int_{\text{surface}} \mathbf{v} \cdot d\mathbf{a} = 16 + 0 + 12 - 12 + 4 = 20.$$

(c) **Volume Integrals.** A volume integral is an expression of the form

$$\int_{\mathcal{V}} T\,d\tau, \tag{1.51}$$

where T is a scalar function and $d\tau$ is an infinitesimal volume element. In Cartesian coordinates,

$$d\tau = dx\,dy\,dz. \tag{1.52}$$

For example, if T is the density of a substance (which might vary from point to point), then the volume integral would give the total mass. Occasionally we shall encounter volume integrals of *vector* functions:

$$\int \mathbf{v}\,d\tau = \int (v_x\,\hat{\mathbf{x}} + v_y\,\hat{\mathbf{y}} + v_z\,\hat{\mathbf{z}})d\tau = \hat{\mathbf{x}} \int v_x d\tau + \hat{\mathbf{y}} \int v_y d\tau + \hat{\mathbf{z}} \int v_z d\tau; \tag{1.53}$$

because the unit vectors ($\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$) are constants, they come outside the integral.

Example 1.8. Calculate the volume integral of $T = xyz^2$ over the prism in Fig. 1.24.

Solution

You can do the three integrals in any order. Let's do x first: it runs from 0 to $(1-y)$, then y (it goes from 0 to 1), and finally z (0 to 3):

$$\int T\,d\tau = \int_0^3 z^2 \left\{ \int_0^1 y \left[\int_0^{1-y} x\,dx \right] dy \right\} dz$$

$$= \frac{1}{2}\int_0^3 z^2\,dz \int_0^1 (1-y)^2 y\,dy = \frac{1}{2}(9)\left(\frac{1}{12}\right) = \frac{3}{8}.$$

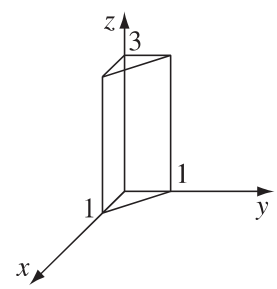


FIGURE 1.24

Problem 1.29 Calculate the line integral of the function $\mathbf{v} = x^2\,\hat{\mathbf{x}} + 2yz\,\hat{\mathbf{y}} + y^2\,\hat{\mathbf{z}}$ from the origin to the point (1,1,1) by three different routes:

(a) $(0, 0, 0) \to (1, 0, 0) \to (1, 1, 0) \to (1, 1, 1)$.

(b) $(0, 0, 0) \to (0, 0, 1) \to (0, 1, 1) \to (1, 1, 1)$.

(c) The direct straight line.

(d) What is the line integral around the closed loop that goes *out* along path (a) and *back* along path (b)?

Problem 1.30 Calculate the surface integral of the function in Ex. 1.7, over the *bottom* of the box. For consistency, let "upward" be the positive direction. Does the surface integral depend only on the boundary line for this function? What is the total flux over the *closed* surface of the box (*including* the bottom)? [*Note:* For the *closed* surface, the positive direction is "outward," and hence "down," for the bottom face.]

Problem 1.31 Calculate the volume integral of the function $T = z^2$ over the tetrahedron with corners at (0,0,0), (1,0,0), (0,1,0), and (0,0,1).

1.3.2 ■ The Fundamental Theorem of Calculus

Suppose $f(x)$ is a function of one variable. The **fundamental theorem of calculus** says:

$$\int_a^b \left(\frac{df}{dx}\right) dx = f(b) - f(a). \tag{1.54}$$

In case this doesn't look familiar, I'll write it another way:

$$\int_a^b F(x)\, dx = f(b) - f(a),$$

where $df/dx = F(x)$. The fundamental theorem tells you how to integrate $F(x)$: you think up a function $f(x)$ whose *derivative* is equal to F.

Geometrical Interpretation: According to Eq. 1.33, $df = (df/dx)dx$ is the infinitesimal change in f when you go from (x) to $(x + dx)$. The fundamental theorem (Eq. 1.54) says that if you chop the interval from a to b (Fig. 1.25) into many tiny pieces, dx, and add up the increments df from each little piece, the result is (not surprisingly) equal to the total change in f: $f(b) - f(a)$. In other words, there are two ways to determine the total change in the function: *either* subtract the values at the ends *or* go step-by-step, adding up all the tiny increments as you go. You'll get the same answer either way.

Notice the basic format of the fundamental theorem: the *integral* of a *derivative* over some *region* is given by the *value of the function* at the end points *(boundaries)*. In vector calculus there are three species of derivative (gradient, divergence, and curl), and each has its own "fundamental theorem," with essentially the same format. I don't plan to prove these theorems here; rather, I will explain what they *mean*, and try to make them *plausible*. Proofs are given in Appendix A.

1.3.3 ■ The Fundamental Theorem for Gradients

Suppose we have a scalar function of three variables $T(x, y, z)$. Starting at point $\mathbf{a}$, we move a small distance $d\mathbf{l}_1$ (Fig. 1.26). According to Eq. 1.37, the function T will change by an amount

$$dT = (\nabla T) \cdot d\mathbf{l}_1.$$

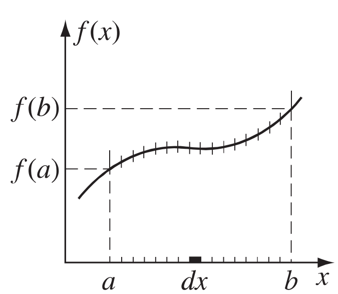


FIGURE 1.25

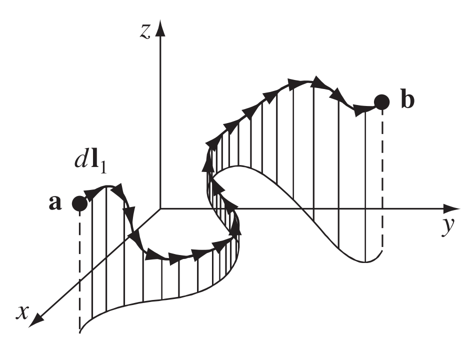


FIGURE 1.26

Now we move a little further, by an additional small displacement $d\mathbf{l}_2$; the incremental change in T will be $(\nabla T)\cdot d\mathbf{l}_2$. In this manner, proceeding by infinitesimal steps, we make the journey to point $\mathbf{b}$. At each step we compute the gradient of T (at that point) and dot it into the displacement $d\mathbf{l}$... this gives us the change in T. Evidently the *total* change in T in going from $\mathbf{a}$ to $\mathbf{b}$ (along the path selected) is

$$\boxed{\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T)\cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}).} \tag{1.55}$$

This is the **fundamental theorem for gradients**; like the "ordinary" fundamental theorem, it says that the integral (here a *line* integral) of a derivative (here the *gradient*) is given by the value of the function at the boundaries ($\mathbf{a}$ and $\mathbf{b}$).

Geometrical Interpretation: Suppose you wanted to determine the height of the Eiffel Tower. You could climb the stairs, using a ruler to measure the rise at each step, and adding them all up (that's the left side of Eq. 1.55), or you could place altimeters at the top and the bottom, and subtract the two readings (that's the right side); you should get the same answer either way (that's the fundamental theorem).

Incidentally, as we found in Ex. 1.6, line integrals ordinarily depend on the *path* taken from $\mathbf{a}$ to $\mathbf{b}$. But the *right* side of Eq. 1.55 makes no reference to the path—only to the end points. Evidently, *gradients* have the special property that their line integrals are path independent:

Corollary 1: $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla T)\cdot d\mathbf{l}$ is independent of the path taken from $\mathbf{a}$ to $\mathbf{b}$.

Corollary 2: $\oint(\nabla T)\cdot d\mathbf{l} = 0$, since the beginning and end points are identical, and hence $T(\mathbf{b}) - T(\mathbf{a}) = 0$.

Example 1.9. Let $T = xy^2$, and take point $\mathbf{a}$ to be the origin $(0, 0, 0)$ and $\mathbf{b}$ the point $(2, 1, 0)$. Check the fundamental theorem for gradients.

Solution

Although the integral is independent of path, we must *pick* a specific path in order to evaluate it. Let's go out along the x axis (step i) and then up (step ii) (Fig. 1.27). As always, $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $\nabla T = y^2\,\hat{\mathbf{x}} + 2xy\,\hat{\mathbf{y}}$.

(i) $y = 0$; $d\mathbf{l} = dx\,\hat{\mathbf{x}}$, $\nabla T\cdot d\mathbf{l} = y^2\,dx = 0$, so

$$\int_{\text{i}} \nabla T\cdot d\mathbf{l} = 0.$$

(ii) $x = 2$; $d\mathbf{l} = dy\,\hat{\mathbf{y}}$, $\nabla T\cdot d\mathbf{l} = 2xy\,dy = 4y\,dy$, so

$$\int_{\text{ii}} \nabla T\cdot d\mathbf{l} = \int_0^1 4y\,dy = 2y^2\Big|_0^1 = 2.$$

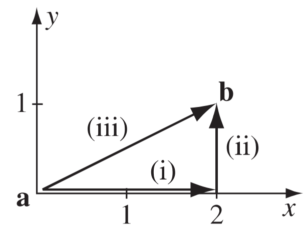


FIGURE 1.27

The total line integral is 2. Is this consistent with the fundamental theorem? Yes: $T(\mathbf{b}) - T(\mathbf{a}) = 2 - 0 = 2$.

Now, just to convince you that the answer is independent of path, let me calculate the same integral along path iii (the straight line from **a** to **b**):

(iii) $y = \frac{1}{2}x,\ dy = \frac{1}{2}\,dx,\ \nabla T \cdot d\mathbf{l} = y^2\,dx + 2xy\,dy = \frac{3}{4}x^2\,dx$, so

$$\int_{\text{iii}} \nabla T \cdot d\mathbf{l} = \int_0^2 \tfrac{3}{4}x^2\,dx = \tfrac{1}{4}x^3\Big|_0^2 = 2.$$

Problem 1.32 Check the fundamental theorem for gradients, using $T = x^2 + 4xy + 2yz^3$, the points $\mathbf{a} = (0, 0, 0)$, $\mathbf{b} = (1, 1, 1)$, and the three paths in Fig. 1.28:

(a) $(0, 0, 0) \to (1, 0, 0) \to (1, 1, 0) \to (1, 1, 1)$;

(b) $(0, 0, 0) \to (0, 0, 1) \to (0, 1, 1) \to (1, 1, 1)$;

(c) the parabolic path $z = x^2$; $y = x$.

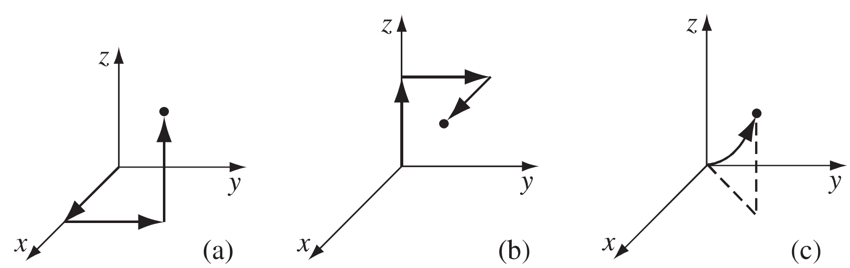


FIGURE 1.28

1.3.4 ■ The Fundamental Theorem for Divergences

The fundamental theorem for divergences states that:

$$\boxed{\int_{\mathcal{V}} (\nabla \cdot \mathbf{v})\,d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a}.} \tag{1.56}$$

In honor, I suppose, of its great importance, this theorem has at least three special names: **Gauss's theorem**, **Green's theorem**, or simply the **divergence theorem**. Like the other "fundamental theorems," it says that the *integral* of a *derivative* (in this case the *divergence*) over a *region* (in this case a *volume*, $\mathcal{V}$) is equal to the value of the function at the *boundary* (in this case the *surface* $\mathcal{S}$ that bounds the volume). Notice that the boundary term is itself an integral (specifically, a surface integral). This is reasonable: the "boundary" of a *line* is just two end points, but the boundary of a *volume* is a (closed) surface.

Geometrical Interpretation: If $\mathbf{v}$ represents the flow of an incompressible fluid, then the *flux* of $\mathbf{v}$ (the right side of Eq. 1.56) is the total amount of fluid passing out through the surface, per unit time. Now, the divergence measures the "spreading out" of the vectors from a point—a place of high divergence is like a "faucet," pouring out liquid. If we have a bunch of faucets in a region filled with incompressible fluid, an equal amount of liquid will be forced out through the boundaries of the region. In fact, there are *two* ways we could determine how much is being produced: (a) we could count up all the faucets, recording how much each puts out, or (b) we could go around the boundary, measuring the flow at each point, and add it all up. You get the same answer either way:

$$\int \text{(faucets within the volume)} = \oint \text{(flow out through the surface)}.$$

This, in essence, is what the divergence theorem says.

Example 1.10. Check the divergence theorem using the function

$$\mathbf{v} = y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + (2yz)\,\hat{\mathbf{z}}$$

and a unit cube at the origin (Fig. 1.29).

Solution

In this case

$$\nabla \cdot \mathbf{v} = 2(x + y),$$

and

$$\int_{\mathcal{V}} 2(x + y)\,d\tau = 2\int_0^1\int_0^1\int_0^1 (x + y)\,dx\,dy\,dz,$$

$$\int_0^1 (x + y)\,dx = \tfrac{1}{2} + y, \quad \int_0^1 (\tfrac{1}{2} + y)\,dy = 1, \quad \int_0^1 1\,dz = 1.$$

Thus,

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{v}\,d\tau = 2.$$

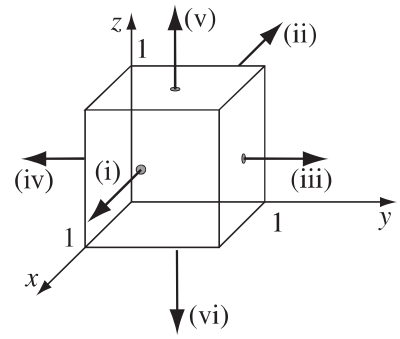


FIGURE 1.29

So much for the left side of the divergence theorem. To evaluate the surface integral we must consider separately the six faces of the cube:

(i) $$\int \mathbf{v} \cdot d\mathbf{a} = \int_0^1 \int_0^1 y^2 dy\, dz = \tfrac{1}{3}.$$

(ii) $$\int \mathbf{v} \cdot d\mathbf{a} = -\int_0^1 \int_0^1 y^2\, dy\, dz = -\tfrac{1}{3}.$$

(iii) $$\int \mathbf{v} \cdot d\mathbf{a} = \int_0^1 \int_0^1 (2x + z^2)\, dx\, dz = \tfrac{4}{3}.$$

(iv) $$\int \mathbf{v} \cdot d\mathbf{a} = -\int_0^1 \int_0^1 z^2\, dx\, dz = -\tfrac{1}{3}.$$

(v) $$\int \mathbf{v} \cdot d\mathbf{a} = \int_0^1 \int_0^1 2y\, dx\, dy = 1.$$

(vi) $$\int \mathbf{v} \cdot d\mathbf{a} = -\int_0^1 \int_0^1 0\, dx\, dy = 0.$$

So the total flux is:

$$\oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a} = \tfrac{1}{3} - \tfrac{1}{3} + \tfrac{4}{3} - \tfrac{1}{3} + 1 + 0 = 2,$$

as expected.

Problem 1.33 Test the divergence theorem for the function $\mathbf{v} = (xy)\,\hat{\mathbf{x}} + (2yz)\,\hat{\mathbf{y}} + (3zx)\,\hat{\mathbf{z}}$. Take as your volume the cube shown in Fig. 1.30, with sides of length 2.

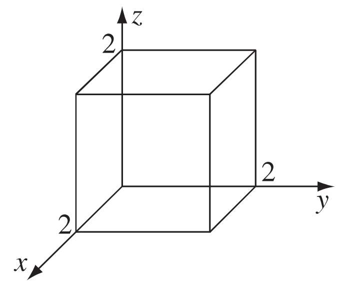


FIGURE 1.30

1.3.5 ■ The Fundamental Theorem for Curls

The fundamental theorem for curls, which goes by the special name of **Stokes' theorem**, states that

$$\int_{\mathcal{S}} (\boldsymbol{\nabla} \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}. \tag{1.57}$$

As always, the *integral* of a *derivative* (here, the *curl*) over a *region* (here, a patch of *surface*, $\mathcal{S}$) is equal to the value of the function at the *boundary* (here, the perimeter of the patch, $\mathcal{P}$). As in the case of the divergence theorem, the boundary term is itself an integral—specifically, a closed line integral.

Geometrical Interpretation: Recall that the curl measures the "twist" of the vectors $\mathbf{v}$; a region of high curl is a whirlpool—if you put a tiny paddle wheel there, it will rotate. Now, the integral of the curl over some surface (or, more precisely, the *flux* of the curl *through* that surface) represents the "total amount of swirl," and we can determine that just as well by going around the edge and finding how much the flow is following the boundary (Fig. 1.31). Indeed, $\oint \mathbf{v} \cdot d\mathbf{l}$ is sometimes called the **circulation** of $\mathbf{v}$.

You may have noticed an apparent ambiguity in Stokes' theorem: concerning the boundary line integral, which *way* are we supposed to go around (clockwise or counterclockwise)? If we go the "wrong" way, we'll pick up an overall sign error. The answer is that it doesn't matter which way you go as long as you are consistent, for there is a compensating sign ambiguity in the surface integral: Which way does $d\mathbf{a}$ point? For a *closed* surface (as in the divergence theorem), $d\mathbf{a}$ points in the direction of the *outward* normal; but for an *open* surface, which way is "out"? Consistency in Stokes' theorem (as in all such matters) is given by the right-hand rule: if your fingers point in the direction of the line integral, then your thumb fixes the direction of $d\mathbf{a}$ (Fig. 1.32).

Now, there are plenty of surfaces (infinitely many) that share any given boundary line. Twist a paper clip into a loop, and dip it in soapy water. The soap film constitutes a surface, with the wire loop as its boundary. If you blow on it, the soap film will expand, making a larger surface, with the same boundary. Ordinarily, a flux integral depends critically on what surface you integrate over, but evidently

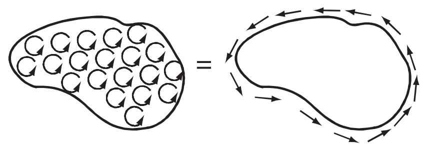


FIGURE 1.31

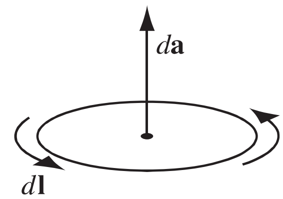


FIGURE 1.32

this is *not* the case with curls. For Stokes' theorem says that $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a}$ is equal to the line integral of $\mathbf{v}$ around the boundary, and the latter makes no reference to the specific surface you choose.

Corollary 1: $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a}$ depends only on the boundary line, not on the particular surface used.

Corollary 2: $\oint (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0$ for any closed surface, since the boundary line, like the mouth of a balloon, shrinks down to a point, and hence the right side of Eq. 1.57 vanishes.

These corollaries are analogous to those for the gradient theorem. We will develop the parallel further in due course.

Example 1.11. Suppose $\mathbf{v} = (2xz + 3y^2)\hat{\mathbf{y}} + (4yz^2)\hat{\mathbf{z}}$. Check Stokes' theorem for the square surface shown in Fig. 1.33.

Solution

Here

$$\nabla \times \mathbf{v} = (4z^2 - 2x)\,\hat{\mathbf{x}} + 2z\,\hat{\mathbf{z}} \quad \text{and} \quad d\mathbf{a} = dy\,dz\,\hat{\mathbf{x}}.$$

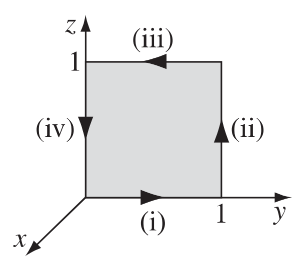


FIGURE 1.33

(In saying that $d\mathbf{a}$ points in the x direction, we are committing ourselves to a counterclockwise line integral. We could as well write $d\mathbf{a} = -dy\,dz\,\hat{\mathbf{x}}$, but then we would be obliged to go clockwise.) Since $x = 0$ for this surface,

$$\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \int_0^1 \int_0^1 4z^2\,dy\,dz = \frac{4}{3}.$$

Now, what about the line integral? We must break this up into four segments:

$$\text{(i)} \quad x = 0, \quad z = 0, \quad \mathbf{v}\cdot d\mathbf{l} = 3y^2\,dy, \quad \int \mathbf{v}\cdot d\mathbf{l} = \int_0^1 3y^2\,dy = 1,$$

$$\text{(ii)} \quad x = 0, \quad y = 1, \quad \mathbf{v}\cdot d\mathbf{l} = 4z^2\,dz, \quad \int \mathbf{v}\cdot d\mathbf{l} = \int_0^1 4z^2\,dz = \frac{4}{3},$$

$$\text{(iii)} \quad x = 0, \quad z = 1, \quad \mathbf{v}\cdot d\mathbf{l} = 3y^2\,dy, \quad \int \mathbf{v}\cdot d\mathbf{l} = \int_1^0 3y^2\,dy = -1,$$

$$\text{(iv)} \quad x = 0, \quad y = 0, \quad \mathbf{v}\cdot d\mathbf{l} = 0, \quad \int \mathbf{v}\cdot d\mathbf{l} = \int_1^0 0\,dz = 0.$$

So

$$\oint \mathbf{v}\cdot d\mathbf{l} = 1 + \frac{4}{3} - 1 + 0 = \frac{4}{3}.$$

It checks.

A point of strategy: notice how I handled step (iii). There is a temptation to write $d\mathbf{l} = -dy\,\hat{\mathbf{y}}$ here, since the path goes to the left. You can get away with this, if you absolutely insist, by running the integral from $0 \to 1$. But it is much safer to say $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$ *always* (never any minus signs) and let the limits of the integral take care of the direction.

Problem 1.34 Test Stokes' theorem for the function $\mathbf{v} = (xy)\,\hat{\mathbf{x}} + (2yz)\,\hat{\mathbf{y}} + (3zx)\,\hat{\mathbf{z}}$, using the triangular shaded area of Fig. 1.34.

Problem 1.35 Check Corollary 1 by using the same function and boundary line as in Ex. 1.11, but integrating over the five faces of the cube in Fig. 1.35. The back of the cube is open.

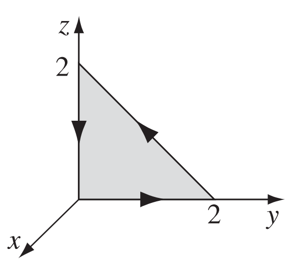


FIGURE 1.34

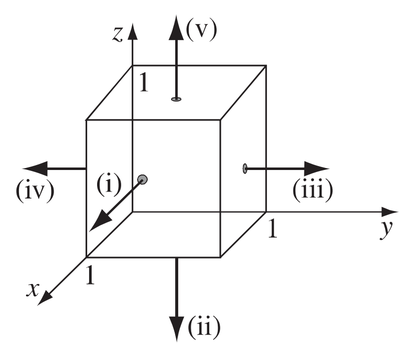


FIGURE 1.35

1.3.6 ■ Integration by Parts

The technique known (awkwardly) as **integration by parts** exploits the product rule for derivatives:

$$\frac{d}{dx}(fg) = f\left(\frac{dg}{dx}\right) + g\left(\frac{df}{dx}\right).$$

Integrating both sides, and invoking the fundamental theorem:

$$\int_a^b \frac{d}{dx}(fg)\,dx = fg\Big|_a^b = \int_a^b f\left(\frac{dg}{dx}\right)dx + \int_a^b g\left(\frac{df}{dx}\right)dx,$$

or

$$\int_a^b f\left(\frac{dg}{dx}\right)dx = -\int_a^b g\left(\frac{df}{dx}\right)dx + fg\Big|_a^b. \tag{1.58}$$

That's integration by parts. It applies to the situation in which you are called upon to integrate the product of one function (f) and the *derivative* of another (g); it says you can *transfer the derivative from g to f*, at the cost of a minus sign and a boundary term.

Example 1.12. Evaluate the integral

$$\int_0^\infty xe^{-x}\,dx.$$

Solution

The exponential can be expressed as a derivative:

$$e^{-x} = \frac{d}{dx}\left(-e^{-x}\right);$$

in this case, then, $f(x) = x$, $g(x) = -e^{-x}$, and $df/dx = 1$, so

$$\int_0^\infty xe^{-x}\,dx = \int_0^\infty e^{-x}\,dx - xe^{-x}\Big|_0^\infty = -e^{-x}\Big|_0^\infty = 1.$$

We can exploit the product rules of vector calculus, together with the appropriate fundamental theorems, in exactly the same way. For example, integrating

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

over a volume, and invoking the divergence theorem, yields

$$\int \nabla\cdot(f\mathbf{A})\,d\tau = \int f(\nabla\cdot\mathbf{A})\,d\tau + \int \mathbf{A}\cdot(\nabla f)\,d\tau = \oint f\mathbf{A}\cdot d\mathbf{a},$$

or

$$\int_{\mathcal{V}} f(\nabla\cdot\mathbf{A})\,d\tau = -\int_{\mathcal{V}} \mathbf{A}\cdot(\nabla f)\,d\tau + \oint_{\mathcal{S}} f\mathbf{A}\cdot d\mathbf{a}. \tag{1.59}$$

Here again the integrand is the product of one function (f) and the derivative (in this case the *divergence*) of another ($\mathbf{A}$), and integration by parts licenses us to transfer the derivative from **A** to f (where it becomes a *gradient*), at the cost of a minus sign and a boundary term (in this case a surface integral).

You might wonder how often one is likely to encounter an integral involving the product of one function and the derivative of another; the answer is *surprisingly* often, and integration by parts turns out to be one of the most powerful tools in vector calculus.

Problem 1.36

(a) Show that

$$\int_{\mathcal{S}} f(\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int_{\mathcal{S}} [\mathbf{A} \times (\nabla f)] \cdot d\mathbf{a} + \oint_{\mathcal{P}} f\mathbf{A} \cdot d\mathbf{l}. \tag{1.60}$$

(b) Show that

$$\int_{\mathcal{V}} \mathbf{B} \cdot (\nabla \times \mathbf{A})\, d\tau = \int_{\mathcal{V}} \mathbf{A} \cdot (\nabla \times \mathbf{B})\, d\tau + \oint_{\mathcal{S}} (\mathbf{A} \times \mathbf{B}) \cdot d\mathbf{a}. \tag{1.61}$$

1.4 ■ CURVILINEAR COORDINATES

1.4.1 ■ Spherical Coordinates

You can label a point P by its Cartesian coordinates (x, y, z), but sometimes it is more convenient to use **spherical** coordinates (r, θ, ϕ); r is the distance from the origin (the magnitude of the position vector **r**), θ (the angle down from the z axis) is called the **polar angle**, and ϕ (the angle around from the x axis) is the **azimuthal angle**. Their relation to Cartesian coordinates can be read from Fig. 1.36:

$$x = r \sin\theta \cos\phi, \qquad y = r \sin\theta \sin\phi, \qquad z = r \cos\theta. \tag{1.62}$$

Figure 1.36 also shows three unit vectors, $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$, pointing in the direction of increase of the corresponding coordinates. They constitute an orthogonal (mutually perpendicular) basis set (just like $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$), and any vector **A** can be expressed in terms of them, in the usual way:

$$\mathbf{A} = A_r\, \hat{\mathbf{r}} + A_\theta\, \hat{\boldsymbol{\theta}} + A_\phi\, \hat{\boldsymbol{\phi}}; \tag{1.63}$$

A_r, A_θ, and A_ϕ are the radial, polar, and azimuthal components of **A**. In terms of the Cartesian unit vectors,

$$\left.\begin{aligned} \hat{\mathbf{r}} &= \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}, \\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}, \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}, \end{aligned}\right\} \tag{1.64}$$

as you can check for yourself (Prob. 1.38). I have put these formulas inside the back cover, for easy reference.

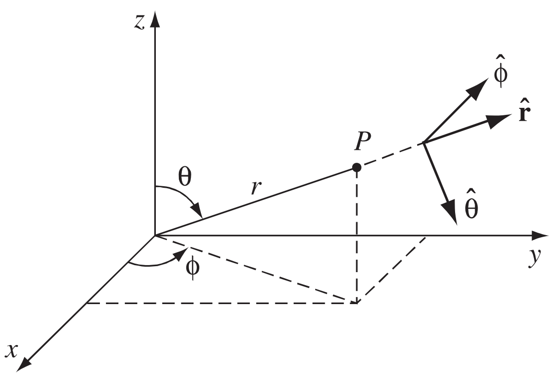


FIGURE 1.36

But there is a poisonous snake lurking here that I'd better warn you about: $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ are associated with a *particular point P*, and they *change direction* as P moves around. For example, $\hat{\mathbf{r}}$ always points radially outward, but "radially outward" can be the x direction, the y direction, or any other direction, depending on where you are. In Fig. 1.37, $\mathbf{A} = \hat{\mathbf{y}}$ and $\mathbf{B} = -\hat{\mathbf{y}}$, and yet *both* of them would be written as $\hat{\mathbf{r}}$ in spherical coordinates. One could take account of this by explicitly indicating the point of reference: $\hat{\mathbf{r}}(\theta, \phi)$, $\hat{\boldsymbol{\theta}}(\theta, \phi)$, $\hat{\boldsymbol{\phi}}(\theta, \phi)$, but this would be cumbersome, and as long as you are alert to the problem, I don't think it will cause difficulties.[9] In particular, do not naïvely combine the spherical components of vectors associated with different points (in Fig. 1.37, $\mathbf{A} + \mathbf{B} = \mathbf{0}$, not $2\hat{\mathbf{r}}$, and $\mathbf{A} \cdot \mathbf{B} = -1$, not +1). Beware of differentiating a vector that is expressed in spherical coordinates, since the unit vectors themselves are functions of position ($\partial\hat{\mathbf{r}}/\partial\theta = \hat{\boldsymbol{\theta}}$, for example). And do not take $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ outside an integral, as I did with $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ in Eq. 1.53. In general, if you're uncertain about the validity of an operation, rewrite the problem using Cartesian coordinates, for which this difficulty does not arise.

An infinitesimal displacement in the $\hat{\mathbf{r}}$ direction is simply dr (Fig. 1.38a), just as an infinitesimal element of length in the x direction is dx:

$$dl_r = dr. \tag{1.65}$$

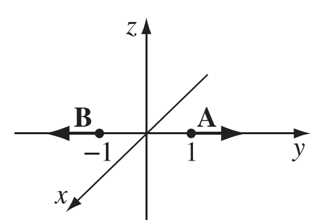


FIGURE 1.37

[9]I claimed back at the beginning that vectors have no location, and I'll stand by that. The vectors themselves live "out there," completely independent of our choice of coordinates. But the *notation* we use to represent them *does* depend on the point in question, in curvilinear coordinates.

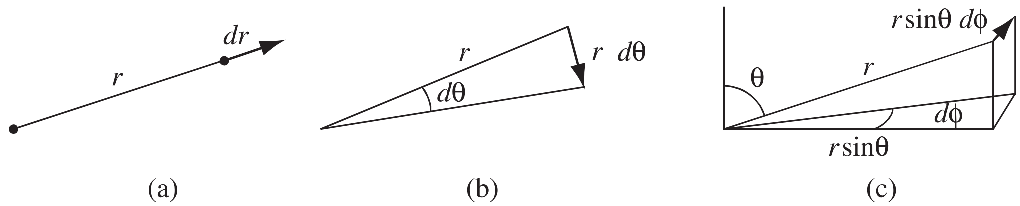


FIGURE 1.38

On the other hand, an infinitesimal element of length in the $\hat{\boldsymbol{\theta}}$ direction (Fig. 1.38b) is *not* just $d\theta$ (that's an *angle*—it doesn't even have the right *units* for a length); rather,

$$dl_\theta = r\,d\theta. \tag{1.66}$$

Similarly, an infinitesimal element of length in the $\hat{\boldsymbol{\phi}}$ direction (Fig. 1.38c) is

$$dl_\phi = r\sin\theta\,d\phi. \tag{1.67}$$

Thus the general infinitesimal displacement $d\mathbf{l}$ is

$$d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}. \tag{1.68}$$

This plays the role (in line integrals, for example) that $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$ played in Cartesian coordinates.

The infinitesimal volume element $d\tau$, in spherical coordinates, is the product of the three infinitesimal displacements:

$$d\tau = dl_r\,dl_\theta\,dl_\phi = r^2\sin\theta\,dr\,d\theta\,d\phi. \tag{1.69}$$

I cannot give you a general expression for *surface* elements $d\mathbf{a}$, since these depend on the orientation of the surface. You simply have to analyze the geometry for any given case (this goes for Cartesian and curvilinear coordinates alike). If you are integrating over the surface of a sphere, for instance, then r is constant, whereas θ and ϕ change (Fig. 1.39), so

$$d\mathbf{a}_1 = dl_\theta\,dl_\phi\,\hat{\mathbf{r}} = r^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}.$$

On the other hand, if the surface lies in the xy plane, say, so that θ is constant (to wit: $\pi/2$) while r and ϕ vary, then

$$d\mathbf{a}_2 = dl_r\,dl_\phi\,\hat{\boldsymbol{\theta}} = r\,dr\,d\phi\,\hat{\boldsymbol{\theta}}.$$

Notice, finally, that r ranges from 0 to ∞, ϕ from 0 to 2π, and θ from 0 to π (*not* 2π—that would count every point twice).[10]

[10] Alternatively, you could run ϕ from 0 to π (the "eastern hemisphere") and cover the "western hemisphere" by extending θ from π up to 2π. But this is very bad notation, since, among other things, $\sin\theta$ will then run negative, and you'll have to put absolute value signs around that term in volume and surface elements (area and volume being intrinsically positive quantities).

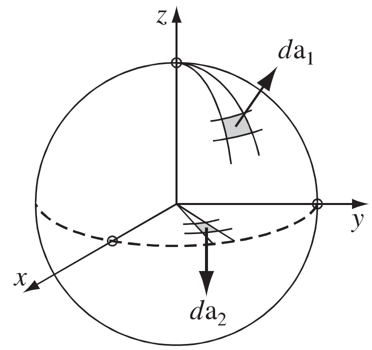


FIGURE 1.39

Example 1.13. Find the volume of a sphere of radius R.

Solution

$$V = \int d\tau = \int_{r=0}^{R} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \left(\int_0^R r^2 \, dr \right) \left(\int_0^{\pi} \sin\theta \, d\theta \right) \left(\int_0^{2\pi} d\phi \right)$$

$$= \left(\frac{R^3}{3} \right) (2)(2\pi) = \frac{4}{3} \pi R^3$$

(not a big surprise).

So far we have talked only about the *geometry* of spherical coordinates. Now I would like to "translate" the vector derivatives (gradient, divergence, curl, and Laplacian) into r, θ, ϕ notation. In principle, this is entirely straightforward: in the case of the gradient,

$$\nabla T = \frac{\partial T}{\partial x} \hat{\mathbf{x}} + \frac{\partial T}{\partial y} \hat{\mathbf{y}} + \frac{\partial T}{\partial z} \hat{\mathbf{z}},$$

for instance, we would first use the chain rule to expand the partials:

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial r} \left(\frac{\partial r}{\partial x} \right) + \frac{\partial T}{\partial \theta} \left(\frac{\partial \theta}{\partial x} \right) + \frac{\partial T}{\partial \phi} \left(\frac{\partial \phi}{\partial x} \right).$$

The terms in parentheses could be worked out from Eq. 1.62—or rather, the *inverse* of those equations (Prob. 1.37). Then we'd do the same for $\partial T/\partial y$ and $\partial T/\partial z$. Finally, we'd substitute in the formulas for $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ in terms of $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ (Prob. 1.38). It would take an hour to figure out the gradient in spherical coordinates by this brute-force method. I suppose this is how it was first done, but there is a much more efficient indirect approach, explained in Appendix A, which has the extra advantage of treating all coordinate systems at once. I described the "straightforward" method only to show you that there is nothing subtle or mysterious about transforming to spherical coordinates: you're expressing the *same quantity* (gradient, divergence, or whatever) in different notation, that's all.

Here, then, are the vector derivatives in spherical coordinates:

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}. \tag{1.70}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}. \tag{1.71}$$

Curl:

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}}$$
$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}. \tag{1.72}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}. \tag{1.73}$$

For reference, these formulas are listed inside the front cover.

Problem 1.37 Find formulas for r, θ, ϕ in terms of x, y, z (the inverse, in other words, of Eq. 1.62).

• **Problem 1.38** Express the unit vectors $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ in terms of $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ (that is, derive Eq. 1.64). Check your answers several ways ($\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} \stackrel{?}{=} 1,\ \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} \stackrel{?}{=} 0,\ \hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}} \stackrel{?}{=} \hat{\boldsymbol{\phi}}, \ldots$). Also work out the inverse formulas, giving $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ in terms of $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ (and θ, ϕ).

• **Problem 1.39**

(a) Check the divergence theorem for the function $\mathbf{v}_1 = r^2\hat{\mathbf{r}}$, using as your volume the sphere of radius R, centered at the origin.

(b) Do the same for $\mathbf{v}_2 = (1/r^2)\hat{\mathbf{r}}$. (If the answer surprises you, look back at Prob. 1.16.)

Problem 1.40 Compute the divergence of the function

$$\mathbf{v} = (r\cos\theta)\,\hat{\mathbf{r}} + (r\sin\theta)\,\hat{\boldsymbol{\theta}} + (r\sin\theta\cos\phi)\,\hat{\boldsymbol{\phi}}.$$

Check the divergence theorem for this function, using as your volume the inverted hemispherical bowl of radius R, resting on the xy plane and centered at the origin (Fig. 1.40).

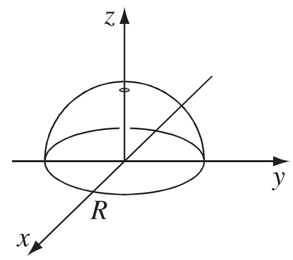


FIGURE 1.40

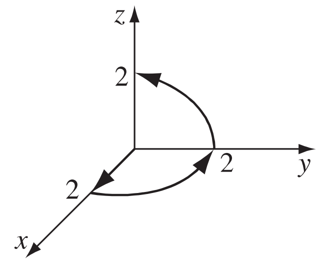


FIGURE 1.41

Problem 1.41 Compute the gradient and Laplacian of the function $T = r(\cos\theta + \sin\theta\cos\phi)$. Check the Laplacian by converting T to Cartesian coordinates and using Eq. 1.42. Test the gradient theorem for this function, using the path shown in Fig. 1.41, from (0, 0, 0) to (0, 0, 2).

1.4.2 ■ Cylindrical Coordinates

The cylindrical coordinates (s, ϕ, z) of a point P are defined in Fig. 1.42. Notice that ϕ has the same meaning as in spherical coordinates, and z is the same as Cartesian; s is the distance to P *from the z axis,* whereas the spherical coordinate r is the distance from the *origin*. The relation to Cartesian coordinates is

$$x = s\cos\phi, \qquad y = s\sin\phi, \qquad z = z. \tag{1.74}$$

The unit vectors (Prob. 1.42) are

$$\left.\begin{array}{rcl} \hat{\mathbf{s}} &=& \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}, \\ \hat{\boldsymbol{\phi}} &=& -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}, \\ \hat{\mathbf{z}} &=& \hat{\mathbf{z}}. \end{array}\right\} \tag{1.75}$$

The infinitesimal displacements are

$$dl_s = ds, \qquad dl_\phi = s\,d\phi, \qquad dl_z = dz, \tag{1.76}$$

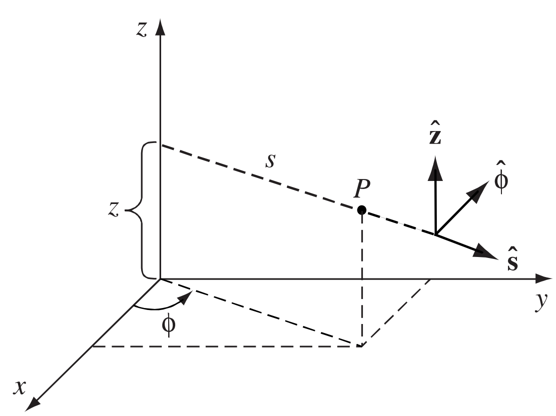


FIGURE 1.42

so

$$d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}, \tag{1.77}$$

and the volume element is

$$d\tau = s\,ds\,d\phi\,dz. \tag{1.78}$$

The range of s is $0 \to \infty$, ϕ goes from $0 \to 2\pi$, and z from $-\infty$ to ∞.
The vector derivatives in cylindrical coordinates are:

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\,\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\,\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\,\hat{\mathbf{z}}. \tag{1.79}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}. \tag{1.80}$$

Curl:

$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}. \tag{1.81}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}. \tag{1.82}$$

These formulas are also listed inside the front cover.

Problem 1.42 Express the cylindrical unit vectors $\hat{\mathbf{s}}$, $\hat{\boldsymbol{\phi}}$, $\hat{\mathbf{z}}$ in terms of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ (that is, derive Eq. 1.75). "Invert" your formulas to get $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ in terms of $\hat{\mathbf{s}}$, $\hat{\boldsymbol{\phi}}$, $\hat{\mathbf{z}}$ (and ϕ).

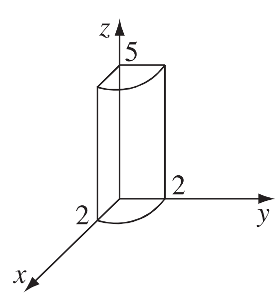


FIGURE 1.43

Problem 1.43

(a) Find the divergence of the function

$$\mathbf{v} = s(2 + \sin^2\phi)\,\hat{\mathbf{s}} + s\sin\phi\cos\phi\;\hat{\boldsymbol{\phi}} + 3z\;\hat{\mathbf{z}}.$$

(b) Test the divergence theorem for this function, using the quarter-cylinder (radius 2, height 5) shown in Fig. 1.43.

(c) Find the curl of **v**.

1.5 ■ THE DIRAC DELTA FUNCTION

1.5.1 ■ The Divergence of $\hat{\mathbf{r}}/r^2$

Consider the vector function

$$\mathbf{v} = \frac{1}{r^2}\,\hat{\mathbf{r}}. \tag{1.83}$$

At every location, **v** is directed radially outward (Fig. 1.44); if ever there was a function that ought to have a large positive divergence, this is it. And yet, when you actually *calculate* the divergence (using Eq. 1.71), you get precisely *zero*:

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0. \tag{1.84}$$

(You will have encountered this paradox already, if you worked Prob. 1.16.) The plot thickens when we apply the divergence theorem to this function. Suppose we integrate over a sphere of radius R, centered at the origin (Prob. 1.38b); the surface integral is

$$\begin{aligned}\oint \mathbf{v}\cdot d\mathbf{a} &= \int\left(\frac{1}{R^2}\hat{\mathbf{r}}\right)\cdot(R^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}})\\ &= \left(\int_0^{\pi}\sin\theta\,d\theta\right)\left(\int_0^{2\pi}d\phi\right) = 4\pi.\end{aligned} \tag{1.85}$$

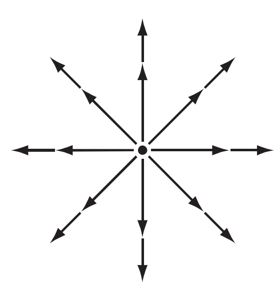


FIGURE 1.44

But the *volume* integral, $\int \nabla \cdot \mathbf{v}\, d\tau$, is *zero*, if we are really to believe Eq. 1.84. Does this mean that the divergence theorem is false? What's going on here?

The source of the problem is the point $r = 0$, where $\mathbf{v}$ blows up (and where, in Eq. 1.84, we have unwittingly divided by zero). It is quite true that $\nabla \cdot \mathbf{v} = 0$ everywhere *except* the origin, but right *at* the origin the situation is more complicated. Notice that the surface integral (Eq. 1.85) is *independent of R*; if the divergence theorem is right (and it *is*), we should get $\int (\nabla \cdot \mathbf{v})\, d\tau = 4\pi$ for *any* sphere centered at the origin, no matter how small. Evidently the entire contribution must be coming from the point $r = 0$! Thus, $\nabla \cdot \mathbf{v}$ has the bizarre property that it vanishes everywhere except at one point, and yet its *integral* (over any volume containing that point) is 4π. No ordinary function behaves like that. (On the other hand, a *physical* example *does* come to mind: the density (mass per unit volume) of a point particle. It's zero except at the exact location of the particle, and yet its *integral* is finite—namely, the mass of the particle.) What we have stumbled on is a mathematical object known to physicists as the **Dirac delta function**. It arises in many branches of theoretical physics. Moreover, the specific problem at hand (the divergence of the function $\hat{\mathbf{r}}/r^2$) is not just some arcane curiosity—it is, in fact, central to the whole theory of electrodynamics. So it is worthwhile to pause here and study the Dirac delta function with some care.

1.5.2 ■ The One-Dimensional Dirac Delta Function

The one-dimensional Dirac delta function, $\delta(x)$, can be pictured as an infinitely high, infinitesimally narrow "spike," with area 1 (Fig. 1.45). That is to say:

$$\delta(x) = \left\{ \begin{array}{ll} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{array} \right\} \tag{1.86}$$

and[11]

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1. \tag{1.87}$$

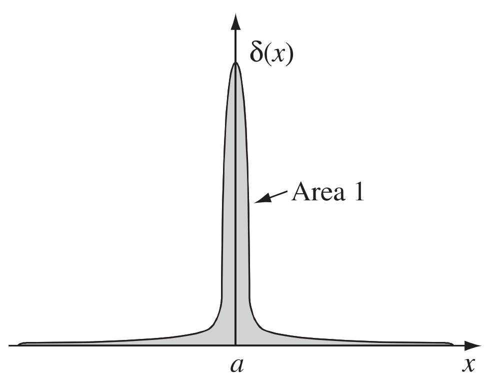


FIGURE 1.45

[11]Notice that the dimensions of $\delta(x)$ are one *over* the dimensions of its argument; if x is a length, $\delta(x)$ carries the units m^{-1}.

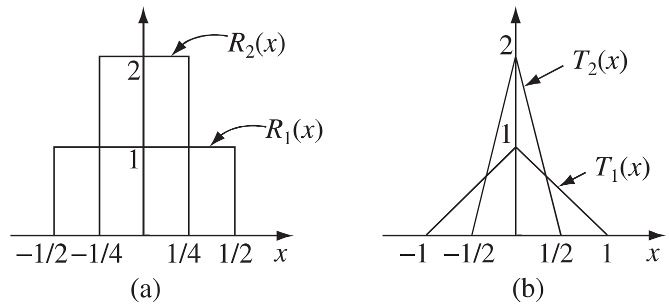


FIGURE 1.46

Technically, $\delta(x)$ is not a function at all, since its value is not finite at $x = 0$; in the mathematical literature it is known as a **generalized function**, or **distribution**. It is, if you like, the *limit* of a *sequence* of functions, such as rectangles $R_n(x)$, of height n and width $1/n$, or isosceles triangles $T_n(x)$, of height n and base $2/n$ (Fig. 1.46).

If $f(x)$ is some "ordinary" function (that is, *not* another delta function—in fact, just to be on the safe side, let's say that $f(x)$ is *continuous*), then the *product* $f(x)\delta(x)$ is zero everywhere except at $x = 0$. It follows that

$$f(x)\delta(x) = f(0)\delta(x). \tag{1.88}$$

(This is the most important fact about the delta function, so make sure you understand why it is true: since the product is zero anyway *except* at $x = 0$, we may as well replace $f(x)$ by the value it assumes at the origin.) In particular

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0)\int_{-\infty}^{\infty} \delta(x)\,dx = f(0). \tag{1.89}$$

Under an integral, then, the delta function "picks out" the value of $f(x)$ at $x = 0$. (Here and below, the integral need not run from $-\infty$ to $+\infty$; it is sufficient that the domain extend across the delta function, and $-\epsilon$ to $+\epsilon$ would do as well.)

Of course, we can shift the spike from $x = 0$ to some other point, $x = a$ (Fig. 1.47):

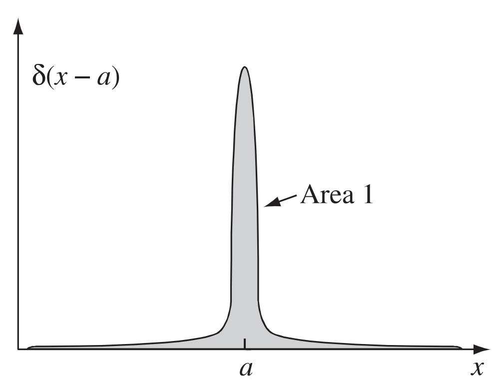


FIGURE 1.47

$$\delta(x-a) = \left\{ \begin{array}{ll} 0, & \text{if } x \neq a \\ \infty, & \text{if } x = a \end{array} \right\} \text{ with } \int_{-\infty}^{\infty} \delta(x-a)\,dx = 1. \tag{1.90}$$

Equation 1.88 becomes

$$f(x)\delta(x-a) = f(a)\delta(x-a), \tag{1.91}$$

and Eq. 1.89 generalizes to

$$\boxed{\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a).} \tag{1.92}$$

Example 1.14. Evaluate the integral

$$\int_0^3 x^3\delta(x-2)\,dx.$$

Solution
The delta function picks out the value of x^3 at the point $x = 2$, so the integral is $2^3 = 8$. Notice, however, that if the upper limit had been 1 (instead of 3), the answer would be 0, because the spike would then be outside the domain of integration.

Although δ itself is not a legitimate function, *integrals* over δ are perfectly acceptable. In fact, it's best to think of the delta function as something that is *always intended for use under an integral sign.* In particular, two expressions involving delta functions (say, $D_1(x)$ and $D_2(x)$) are considered equal if [12]

$$\int_{-\infty}^{\infty} f(x)D_1(x)\,dx = \int_{-\infty}^{\infty} f(x)D_2(x)\,dx, \tag{1.93}$$

for all ("ordinary") functions $f(x)$.

Example 1.15. Show that

$$\delta(kx) = \frac{1}{|k|}\delta(x), \tag{1.94}$$

where k is any (nonzero) constant. (In particular, $\delta(-x) = \delta(x)$.)

[12] I emphasize that the integrals must be equal for *any* $f(x)$. Suppose $D_1(x)$ and $D_2(x)$ actually *differed,* say, in the neighborhood of the point $x = 17$. Then we could pick a function $f(x)$ that was sharply peaked about $x = 17$, and the integrals would not be equal.

Solution

For an arbitrary test function $f(x)$, consider the integral

$$\int_{-\infty}^{\infty} f(x)\delta(kx)\,dx.$$

Changing variables, we let $y \equiv kx$, so that $x = y/k$, and $dx = 1/k\,dy$. If k is positive, the integration still runs from $-\infty$ to $+\infty$, but if k is *negative,* then $x = \infty$ implies $y = -\infty$, and vice versa, so the order of the limits is reversed. Restoring the "proper" order costs a minus sign. Thus

$$\int_{-\infty}^{\infty} f(x)\delta(kx)\,dx = \pm\int_{-\infty}^{\infty} f(y/k)\delta(y)\frac{dy}{k} = \pm\frac{1}{k}f(0) = \frac{1}{|k|}f(0).$$

(The lower signs apply when k is negative, and we account for this neatly by putting absolute value bars around the final k, as indicated.) Under the integral sign, then, $\delta(kx)$ serves the same purpose as $(1/|k|)\delta(x)$:

$$\int_{-\infty}^{\infty} f(x)\delta(kx)\,dx = \int_{-\infty}^{\infty} f(x)\left[\frac{1}{|k|}\delta(x)\right]dx.$$

According to the criterion Eq. 1.93, therefore, $\delta(kx)$ and $(1/|k|)\delta(x)$ are equal.

Problem 1.44 Evaluate the following integrals:

(a) $\int_2^6 (3x^2 - 2x - 1)\,\delta(x-3)\,dx$.

(b) $\int_0^5 \cos x\,\delta(x-\pi)\,dx$.

(c) $\int_0^3 x^3\delta(x+1)\,dx$.

(d) $\int_{-\infty}^{\infty} \ln(x+3)\,\delta(x+2)\,dx$.

Problem 1.45 Evaluate the following integrals:

(a) $\int_{-2}^{2} (2x+3)\,\delta(3x)\,dx$.

(b) $\int_0^2 (x^3 + 3x + 2)\,\delta(1-x)\,dx$.

(c) $\int_{-1}^{1} 9x^2\delta(3x+1)\,dx$.

(d) $\int_{-\infty}^{a} \delta(x-b)\,dx$.

Problem 1.46

(a) Show that

$$x\frac{d}{dx}(\delta(x)) = -\delta(x).$$

[*Hint:* Use integration by parts.]

(b) Let $\theta(x)$ be the **step function**:

$$\theta(x) \equiv \left\{ \begin{array}{ll} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{array} \right\}. \tag{1.95}$$

Show that $d\theta/dx = \delta(x)$.

1.5.3 ■ The Three-Dimensional Delta Function

It is easy to generalize the delta function to three dimensions:

$$\delta^3(\mathbf{r}) = \delta(x)\,\delta(y)\,\delta(z). \tag{1.96}$$

(As always, $\mathbf{r} \equiv x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$ is the position vector, extending from the origin to the point (x, y, z).) This three-dimensional delta function is zero everywhere except at (0, 0, 0), where it blows up. Its volume integral is 1:

$$\int_{\text{all space}} \delta^3(\mathbf{r})\,d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\,\delta(y)\,\delta(z)\,dx\,dy\,dz = 1. \tag{1.97}$$

And, generalizing Eq. 1.92,

$$\int_{\text{all space}} f(\mathbf{r})\delta^3(\mathbf{r}-\mathbf{a})\,d\tau = f(\mathbf{a}). \tag{1.98}$$

As in the one-dimensional case, integration with δ picks out the value of the function f at the location of the spike.

We are now in a position to resolve the paradox introduced in Sect. 1.5.1. As you will recall, we found that the divergence of $\hat{\mathbf{r}}/r^2$ is zero everywhere except at the origin, and yet its *integral* over any volume containing the origin is a constant (to wit: 4π). These are precisely the defining conditions for the Dirac delta function; evidently

$$\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\delta^3(\mathbf{r}). \tag{1.99}$$

More generally,

$$\boxed{\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}}),} \tag{1.100}$$

where, as always, $\boldsymbol{\mathscr{r}}$ is the separation vector: $\boldsymbol{\mathscr{r}} \equiv \mathbf{r} - \mathbf{r}'$. Note that differentiation here is with respect to $\mathbf{r}$, while $\mathbf{r}'$ is held constant. Incidentally, since

$$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \tag{1.101}$$

(Prob. 1.13b), it follows that

$$\nabla^2 \frac{1}{\mathscr{r}} = -4\pi\delta^3(\boldsymbol{\mathscr{r}}). \tag{1.102}$$

Example 1.16. Evaluate the integral

$$J = \int_{\mathcal{V}} (r^2 + 2)\, \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) d\tau,$$

where $\mathcal{V}$ is a sphere[13] of radius R centered at the origin.

Solution 1

Use Eq. 1.99 to rewrite the divergence, and Eq. 1.98 to do the integral:

$$J = \int_{\mathcal{V}} (r^2 + 2) 4\pi\delta^3(\mathbf{r})\, d\tau = 4\pi(0 + 2) = 8\pi.$$

This one-line solution demonstrates something of the power and beauty of the delta function, but I would like to show you a second method, which is much more cumbersome but serves to illustrate the method of integration by parts (Sect. 1.3.6).

Solution 2

Using Eq. 1.59, we transfer the derivative from $\hat{\mathbf{r}}/r^2$ to $(r^2 + 2)$:

$$J = -\int_{\mathcal{V}} \frac{\hat{\mathbf{r}}}{r^2} \cdot [\nabla(r^2 + 2)]\, d\tau + \oint_{\mathcal{S}} (r^2 + 2) \frac{\hat{\mathbf{r}}}{r^2} \cdot d\mathbf{a}.$$

The gradient is

$$\nabla(r^2 + 2) = 2r\hat{\mathbf{r}},$$

so the volume integral becomes

$$\int \frac{2}{r}\, d\tau = \int \frac{2}{r} r^2 \sin\theta\, dr\, d\theta\, d\phi = 8\pi \int_0^R r\, dr = 4\pi R^2.$$

Meanwhile, on the boundary of the sphere (where $r = R$),

$$d\mathbf{a} = R^2 \sin\theta\, d\theta\, d\phi\, \hat{\mathbf{r}},$$

so the surface integral is

$$\int (R^2 + 2) \sin\theta\, d\theta\, d\phi = 4\pi(R^2 + 2).$$

[13] In proper mathematical jargon, "sphere" denotes the *surface*, and "ball" the volume it encloses. But physicists are (as usual) sloppy about this sort of thing, and I use the word "sphere" for both the surface and the volume. Where the meaning is not clear from the context, I will write "spherical surface" or "spherical volume." The language police tell me that the former is redundant and the latter an oxymoron, but a poll of my physics colleagues reveals that this is (for us) the standard usage.

Putting it all together,

$$J = -4\pi R^2 + 4\pi(R^2 + 2) = 8\pi,$$

as before.

Problem 1.47

(a) Write an expression for the volume charge density $\rho(\mathbf{r})$ of a point charge q at $\mathbf{r}'$. Make sure that the volume integral of ρ equals q.

(b) What is the volume charge density of an electric dipole, consisting of a point charge $-q$ at the origin and a point charge $+q$ at $\mathbf{a}$?

(c) What is the volume charge density (in spherical coordinates) of a uniform, infinitesimally thin spherical shell of radius R and total charge Q, centered at the origin? [*Beware:* the integral over all space must equal Q.]

Problem 1.48 Evaluate the following integrals:

(a) $\int (r^2 + \mathbf{r}\cdot\mathbf{a} + a^2)\delta^3(\mathbf{r} - \mathbf{a})\, d\tau$, where $\mathbf{a}$ is a fixed vector, a is its magnitude, and the integral is over all space.

(b) $\int_{\mathcal{V}} |\mathbf{r} - \mathbf{b}|^2 \delta^3(5\mathbf{r})\, d\tau$, where $\mathcal{V}$ is a cube of side 2, centered on the origin, and $\mathbf{b} = 4\,\hat{\mathbf{y}} + 3\,\hat{\mathbf{z}}$.

(c) $\int_{\mathcal{V}} \left[r^4 + r^2(\mathbf{r}\cdot\mathbf{c}) + c^4\right]\delta^3(\mathbf{r} - \mathbf{c})\, d\tau$, where $\mathcal{V}$ is a sphere of radius 6 about the origin, $\mathbf{c} = 5\,\hat{\mathbf{x}} + 3\,\hat{\mathbf{y}} + 2\,\hat{\mathbf{z}}$, and c is its magnitude.

(d) $\int_{\mathcal{V}} \mathbf{r}\cdot(\mathbf{d} - \mathbf{r})\delta^3(\mathbf{e} - \mathbf{r})\, d\tau$, where $\mathbf{d} = (1, 2, 3)$, $\mathbf{e} = (3, 2, 1)$, and $\mathcal{V}$ is a sphere of radius 1.5 centered at (2, 2, 2).

Problem 1.49 Evaluate the integral

$$J = \int_{\mathcal{V}} e^{-r}\left(\nabla\cdot\frac{\hat{\mathbf{r}}}{r^2}\right) d\tau$$

(where $\mathcal{V}$ is a sphere of radius R, centered at the origin) by two different methods, as in Ex. 1.16.

1.6 ■ THE THEORY OF VECTOR FIELDS

1.6.1 ■ The Helmholtz Theorem

Ever since Faraday, the laws of electricity and magnetism have been expressed in terms of **electric** and **magnetic fields**, **E** and **B**. Like many physical laws, these are most compactly expressed as differential equations. Since **E** and **B** are *vectors*, the differential equations naturally involve vector derivatives: divergence and curl. Indeed, Maxwell reduced the entire theory to four equations, specifying respectively the divergence and the curl of **E** and **B**.

Maxwell's formulation raises an important mathematical question: To what extent is a vector function determined by its divergence and curl? In other words, if I tell you that the *divergence* of **F** (which stands for **E** or **B**, as the case may be) is a specified (scalar) function D,

$$\nabla \cdot \mathbf{F} = D,$$

and the curl of **F** is a specified (vector) function **C**,

$$\nabla \times \mathbf{F} = \mathbf{C},$$

(for consistency, **C** must be divergenceless,

$$\nabla \cdot \mathbf{C} = 0,$$

because the divergence of a curl is always zero), can you then determine the function **F**?

Well... not quite. For example, as you may have discovered in Prob. 1.20, there are many functions whose divergence and curl are both zero everywhere—the trivial case $\mathbf{F} = \mathbf{0}$, of course, but also $\mathbf{F} = yz\,\hat{\mathbf{x}} + zx\,\hat{\mathbf{y}} + xy\,\hat{\mathbf{z}}$, $\mathbf{F} = \sin x \cosh y\,\hat{\mathbf{x}} - \cos x \sinh y\,\hat{\mathbf{y}}$, etc. To solve a differential equation you must also be supplied with appropriate **boundary conditions**. In electrodynamics we typically require that the fields go to zero "at infinity" (far away from all charges).[14] With that extra information, the **Helmholtz theorem** guarantees that the field is uniquely determined by its divergence and curl. (The Helmholtz theorem is discussed in Appendix B.)

1.6.2 ■ Potentials

If the curl of a vector field (**F**) vanishes (everywhere), then **F** can be written as the gradient of a **scalar potential** (V):

$$\nabla \times \mathbf{F} = \mathbf{0} \Longleftrightarrow \mathbf{F} = -\nabla V. \qquad (1.103)$$

(The minus sign is purely conventional.) That's the essential burden of the following theorem:

Theorem 1

Curl-less (or "**irrotational**") **fields**. The following conditions are equivalent (that is, **F** satisfies one if and only if it satisfies all the others):

[14] In some textbook problems the charge itself extends to infinity (we speak, for instance, of the electric field of an infinite plane, or the magnetic field of an infinite wire). In such cases the normal boundary conditions do not apply, and one must invoke symmetry arguments to determine the fields uniquely.

(a) $\nabla \times \mathbf{F} = \mathbf{0}$ everywhere.

(b) $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l}$ is independent of path, for any given end points.

(c) $\oint \mathbf{F} \cdot d\mathbf{l} = 0$ for any closed loop.

(d) $\mathbf{F}$ is the gradient of some scalar function: $\mathbf{F} = -\nabla V$.

The potential is not unique—any constant can be added to V with impunity, since this will not affect its gradient.

If the divergence of a vector field ($\mathbf{F}$) vanishes (everywhere), then $\mathbf{F}$ can be expressed as the curl of a **vector potential** ($\mathbf{A}$):

$$\nabla \cdot \mathbf{F} = 0 \Longleftrightarrow \mathbf{F} = \nabla \times \mathbf{A}. \tag{1.104}$$

That's the main conclusion of the following theorem:

Theorem 2

Divergence-less (or "**solenoidal**") **fields**. The following conditions are equivalent:

(a) $\nabla \cdot \mathbf{F} = 0$ everywhere.

(b) $\int \mathbf{F} \cdot d\mathbf{a}$ is independent of surface, for any given boundary line.

(c) $\oint \mathbf{F} \cdot d\mathbf{a} = 0$ for any closed surface.

(d) $\mathbf{F}$ is the curl of some vector function: $\mathbf{F} = \nabla \times \mathbf{A}$.

The vector potential is not unique—the gradient of any scalar function can be added to $\mathbf{A}$ without affecting the curl, since the curl of a gradient is zero.

You should by now be able to prove all the connections in these theorems, save for the ones that say (a), (b), or (c) implies (d). Those are more subtle, and will come later. Incidentally, in *all* cases (*whatever* its curl and divergence may be) a vector field $\mathbf{F}$ can be written as the gradient of a scalar plus the curl of a vector:[15]

$$\mathbf{F} = -\nabla V + \nabla \times \mathbf{A} \qquad \text{(always)}. \tag{1.105}$$

Problem 1.50

(a) Let $\mathbf{F}_1 = x^2\,\hat{\mathbf{z}}$ and $\mathbf{F}_2 = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$. Calculate the divergence and curl of $\mathbf{F}_1$ and $\mathbf{F}_2$. Which one can be written as the gradient of a scalar? Find a scalar potential that does the job. Which one can be written as the curl of a vector? Find a suitable vector potential.

[15]In physics, the word **field** denotes generically any function of position (x, y, z) and time (t). But in electrodynamics two particular fields ($\mathbf{E}$ and $\mathbf{B}$) are of such paramount importance as to preempt the term. Thus technically the potentials are also "fields," but we never call them that.

(b) Show that $\mathbf{F}_3 = yz\,\hat{\mathbf{x}} + zx\,\hat{\mathbf{y}} + xy\,\hat{\mathbf{z}}$ can be written both as the gradient of a scalar and as the curl of a vector. Find scalar and vector potentials for this function.

Problem 1.51 For Theorem 1, show that (d) $\Rightarrow$ (a), (a) $\Rightarrow$ (c), (c) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), and (c) $\Rightarrow$ (a).

Problem 1.52 For Theorem 2, show that (d) $\Rightarrow$ (a), (a) $\Rightarrow$ (c), (c) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), and (c) $\Rightarrow$ (a).

Problem 1.53

(a) Which of the vectors in Problem 1.15 can be expressed as the gradient of a scalar? Find a scalar function that does the job.

(b) Which can be expressed as the curl of a vector? Find such a vector.

More Problems on Chapter 1

Problem 1.54 Check the divergence theorem for the function

$$\mathbf{v} = r^2\cos\theta\,\hat{\mathbf{r}} + r^2\cos\phi\,\hat{\boldsymbol{\theta}} - r^2\cos\theta\sin\phi\,\hat{\boldsymbol{\phi}},$$

using as your volume one octant of the sphere of radius R (Fig. 1.48). Make sure you include the *entire* surface. [*Answer:* $\pi R^4/4$]

Problem 1.55 Check Stokes' theorem using the function $\mathbf{v} = ay\,\hat{\mathbf{x}} + bx\,\hat{\mathbf{y}}$ (a and b are constants) and the circular path of radius R, centered at the origin in the xy plane. [*Answer:* $\pi R^2(b-a)$]

Problem 1.56 Compute the line integral of

$$\mathbf{v} = 6\,\hat{\mathbf{x}} + yz^2\,\hat{\mathbf{y}} + (3y+z)\,\hat{\mathbf{z}}$$

along the triangular path shown in Fig. 1.49. Check your answer using Stokes' theorem. [*Answer:* 8/3]

Problem 1.57 Compute the line integral of

$$\mathbf{v} = (r\cos^2\theta)\,\hat{\mathbf{r}} - (r\cos\theta\sin\theta)\,\hat{\boldsymbol{\theta}} + 3r\,\hat{\boldsymbol{\phi}}$$

around the path shown in Fig. 1.50 (the points are labeled by their Cartesian coordinates). Do it either in cylindrical or in spherical coordinates. Check your answer, using Stokes' theorem. [*Answer:* $3\pi/2$]

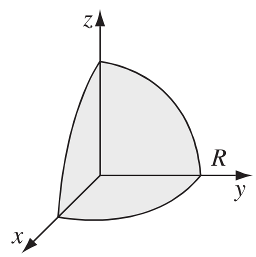


FIGURE 1.48

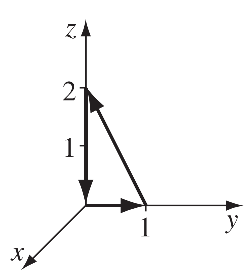


FIGURE 1.49

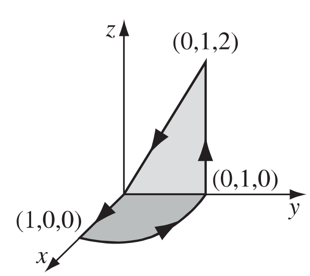


FIGURE 1.50

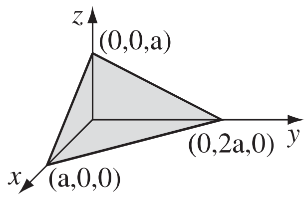


FIGURE 1.51

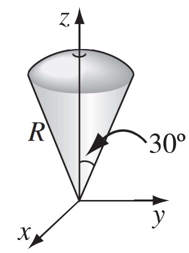


FIGURE 1.52

Problem 1.58 Check Stokes' theorem for the function $\mathbf{v} = y\,\hat{\mathbf{z}}$, using the triangular surface shown in Fig. 1.51. [*Answer:* a^2]

Problem 1.59 Check the divergence theorem for the function

$$\mathbf{v} = r^2 \sin\theta\,\hat{\mathbf{r}} + 4r^2\cos\theta\,\hat{\boldsymbol{\theta}} + r^2\tan\theta\,\hat{\boldsymbol{\phi}},$$

using the volume of the "ice-cream cone" shown in Fig. 1.52 (the top surface is spherical, with radius R and centered at the origin). [*Answer:* $(\pi R^4/12)(2\pi + 3\sqrt{3})$]

Problem 1.60 Here are two cute checks of the fundamental theorems:

(a) Combine Corollary 2 to the gradient theorem with Stokes' theorem ($\mathbf{v} = \boldsymbol{\nabla} T$, in this case). Show that the result is consistent with what you already knew about second derivatives.

(b) Combine Corollary 2 to Stokes' theorem with the divergence theorem. Show that the result is consistent with what you already knew.

• **Problem 1.61** Although the gradient, divergence, and curl theorems are the fundamental integral theorems of vector calculus, it is possible to derive a number of corollaries from them. Show that:

(a) $\int_{\mathcal{V}} (\boldsymbol{\nabla} T)\,d\tau = \oint_{\mathcal{S}} T\,d\mathbf{a}$. [*Hint:* Let $\mathbf{v} = \mathbf{c}T$, where $\mathbf{c}$ is a constant, in the divergence theorem; use the product rules.]

(b) $\int_{\mathcal{V}} (\boldsymbol{\nabla} \times \mathbf{v})\,d\tau = -\oint_{\mathcal{S}} \mathbf{v} \times d\mathbf{a}$. [*Hint:* Replace $\mathbf{v}$ by $(\mathbf{v} \times \mathbf{c})$ in the divergence theorem.]

(c) $\int_{\mathcal{V}} [T\nabla^2 U + (\boldsymbol{\nabla} T)\cdot(\boldsymbol{\nabla} U)]\,d\tau = \oint_{\mathcal{S}} (T\boldsymbol{\nabla} U)\cdot d\mathbf{a}$. [*Hint:* Let $\mathbf{v} = T\boldsymbol{\nabla} U$ in the divergence theorem.]

(d) $\int_{\mathcal{V}} (T\nabla^2 U - U\nabla^2 T)\,d\tau = \oint_{\mathcal{S}} (T\boldsymbol{\nabla} U - U\boldsymbol{\nabla} T)\cdot d\mathbf{a}$. [*Comment:* This is sometimes called **Green's second identity**; it follows from (c), which is known as **Green's identity**.]

(e) $\int_{\mathcal{S}} \boldsymbol{\nabla} T \times d\mathbf{a} = -\oint_{\mathcal{P}} T\,d\mathbf{l}$. [*Hint:* Let $\mathbf{v} = \mathbf{c}T$ in Stokes' theorem.]

• **Problem 1.62** The integral

$$\mathbf{a} \equiv \int_{\mathcal{S}} d\mathbf{a} \tag{1.106}$$

is sometimes called the **vector area** of the surface $\mathcal{S}$. If $\mathcal{S}$ happens to be *flat*, then $|\mathbf{a}|$ is the *ordinary* (scalar) area, obviously.

(a) Find the vector area of a hemispherical bowl of radius R.

(b) Show that $\mathbf{a} = \mathbf{0}$ for any *closed* surface. [*Hint:* Use Prob. 1.61a.]

(c) Show that $\mathbf{a}$ is the same for all surfaces sharing the same boundary.

(d) Show that

$$\mathbf{a} = \frac{1}{2}\oint \mathbf{r} \times d\mathbf{l}, \tag{1.107}$$

where the integral is around the boundary line. [*Hint:* One way to do it is to draw the cone subtended by the loop at the origin. Divide the conical surface up into infinitesimal triangular wedges, each with vertex at the origin and opposite side $d\mathbf{l}$, and exploit the geometrical interpretation of the cross product (Fig. 1.8).]

(e) Show that

$$\oint (\mathbf{c} \cdot \mathbf{r})\, d\mathbf{l} = \mathbf{a} \times \mathbf{c}, \tag{1.108}$$

for any constant vector $\mathbf{c}$. [*Hint:* Let $T = \mathbf{c} \cdot \mathbf{r}$ in Prob. 1.61e.]

• **Problem 1.63**

(a) Find the divergence of the function

$$\mathbf{v} = \frac{\hat{\mathbf{r}}}{r}.$$

First compute it directly, as in Eq. 1.84. Test your result using the divergence theorem, as in Eq. 1.85. Is there a delta function at the origin, as there was for $\hat{\mathbf{r}}/r^2$? What is the general formula for the divergence of $r^n\hat{\mathbf{r}}$? [*Answer:* $\nabla \cdot (r^n\hat{\mathbf{r}}) = (n+2)r^{n-1}$, unless $n = -2$, in which case it is $4\pi\delta^3(\mathbf{r})$; for $n < -2$, the divergence is ill-defined at the origin.]

(b) Find the *curl* of $r^n\hat{\mathbf{r}}$. Test your conclusion using Prob. 1.61b. [*Answer:* $\nabla \times (r^n\hat{\mathbf{r}}) = \mathbf{0}$]

Problem 1.64 In case you're not persuaded that $\nabla^2(1/r) = -4\pi\delta^3(\mathbf{r})$ (Eq. 1.102 with $\mathbf{r}' = \mathbf{0}$ for simplicity), try replacing r by $\sqrt{r^2+\epsilon^2}$, and watching what happens as $\epsilon \to 0$.[16] Specifically, let

$$D(r,\epsilon) \equiv -\frac{1}{4\pi}\nabla^2\frac{1}{\sqrt{r^2+\epsilon^2}}.$$

[16] This problem was suggested by Frederick Strauch.

To demonstrate that this goes to $\delta^3(\mathbf{r})$ as $\epsilon \to 0$:

(a) Show that $D(r, \epsilon) = (3\epsilon^2/4\pi)(r^2 + \epsilon^2)^{-5/2}$.

(b) Check that $D(0, \epsilon) \to \infty$, as $\epsilon \to 0$.

(c) Check that $D(r, \epsilon) \to 0$, as $\epsilon \to 0$, for all $r \neq 0$.

(d) Check that the integral of $D(r, \epsilon)$ over all space is 1.

CHAPTER

7 Electrodynamics

7.1 ■ ELECTROMOTIVE FORCE

7.1.1 ■ Ohm's Law

To make a current flow, you have to *push* on the charges. How *fast* they move, in response to a given push, depends on the nature of the material. For most substances, the current density $\mathbf{J}$ is proportional to the *force per unit charge*, $\mathbf{f}$:

$$\mathbf{J} = \sigma \mathbf{f}. \tag{7.1}$$

The proportionality factor σ (not to be confused with surface charge) is an empirical constant that varies from one material to another; it's called the **conductivity** of the medium. Actually, the handbooks usually list the *reciprocal* of σ, called the **resistivity**: $\rho = 1/\sigma$ (not to be confused with charge density—I'm sorry, but we're running out of Greek letters, and this is the standard notation). Some typical values are listed in Table 7.1. Notice that even *insulators* conduct slightly, though the conductivity of a metal is astronomically greater; in fact, for most purposes metals can be regarded as **perfect conductors**, with $\sigma = \infty$, while for insulators we can pretend $\sigma = 0$.

In principle, the force that drives the charges to produce the current could be anything—chemical, gravitational, or trained ants with tiny harnesses. For *our* purposes, though, it's usually an electromagnetic force that does the job. In this case Eq. 7.1 becomes

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{7.2}$$

Ordinarily, the velocity of the charges is sufficiently small that the second term can be ignored:

$$\boxed{\mathbf{J} = \sigma \mathbf{E}.} \tag{7.3}$$

(However, in plasmas, for instance, the magnetic contribution to $\mathbf{f}$ can be significant.) Equation 7.3 is called **Ohm's law**, though the physics behind it is really contained in Eq. 7.1, of which 7.3 is just a special case.

I know: you're confused because I said $\mathbf{E} = \mathbf{0}$ inside a conductor (Sect. 2.5.1). But that's for *stationary* charges ($\mathbf{J} = \mathbf{0}$). Moreover, for *perfect* conductors

Material	Resistivity	Material	Resistivity
Conductors:		*Semiconductors:*	
Silver	1.59×10^{-8}	Sea water	0.2
Copper	1.68×10^{-8}	Germanium	0.46
Gold	2.21×10^{-8}	Diamond	2.7
Aluminum	2.65×10^{-8}	Silicon	2500
Iron	9.61×10^{-8}	*Insulators:*	
Mercury	9.61×10^{-7}	Water (pure)	8.3×10^{3}
Nichrome	1.08×10^{-6}	Glass	$10^{9} - 10^{14}$
Manganese	1.44×10^{-6}	Rubber	$10^{13} - 10^{15}$
Graphite	1.6×10^{-5}	Teflon	$10^{22} - 10^{24}$

TABLE 7.1 Resistivities, in ohm-meters (all values are for 1 atm, 20° C). *Data from Handbook of Chemistry and Physics,* 91st ed. (Boca Raton, Fla.: CRC Press, 2010) and other references.

$\mathbf{E} = \mathbf{J}/\sigma = \mathbf{0}$ even if current *is* flowing. In practice, metals are such good conductors that the electric field required to drive current in them is negligible. Thus we routinely treat the connecting wires in electric circuits (for example) as equipotentials. **Resistors**, by contrast, are made from *poorly* conducting materials.

Example 7.1. A cylindrical resistor of cross-sectional area A and length L is made from material with conductivity σ. (See Fig. 7.1; as indicated, the cross section need not be circular, but I *do* assume it is the same all the way down.) If we stipulate that the potential is constant over each end, and the potential difference between the ends is V, what current flows?

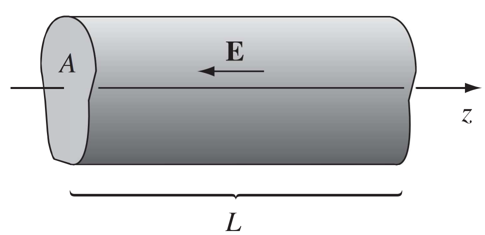


FIGURE 7.1

Solution

As it turns out, the electric field is *uniform* within the wire (I'll prove this in a moment). It follows from Eq. 7.3 that the current density is also uniform, so

$$I = JA = \sigma EA = \frac{\sigma A}{L} V.$$

Example 7.2. Two long coaxial metal cylinders (radii a and b) are separated by material of conductivity σ (Fig. 7.2). If they are maintained at a potential difference V, what current flows from one to the other, in a length L?

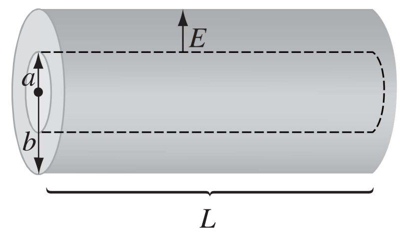


FIGURE 7.2

Solution

The field between the cylinders is

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}},$$

where λ is the charge per unit length on the inner cylinder. The current is therefore

$$I = \int \mathbf{J}\cdot d\mathbf{a} = \sigma\int \mathbf{E}\cdot d\mathbf{a} = \frac{\sigma}{\epsilon_0}\lambda L.$$

(The integral is over any surface enclosing the inner cylinder.) Meanwhile, the potential difference between the cylinders is

$$V = -\int_b^a \mathbf{E}\cdot d\mathbf{l} = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right),$$

so

$$I = \frac{2\pi\sigma L}{\ln(b/a)}V.$$

As these examples illustrate, the total current flowing from one **electrode** to the other is proportional to the potential difference between them:

$$\boxed{V = IR.} \tag{7.4}$$

This, of course, is the more familiar version of Ohm's law. The constant of proportionality R is called the **resistance**; it's a function of the geometry of the arrangement and the conductivity of the medium between the electrodes. (In Ex. 7.1, $R = (L/\sigma A)$; in Ex. 7.2, $R = \ln(b/a)/2\pi\sigma L$.) Resistance is measured in **ohms** (Ω): an ohm is a volt per ampere. Notice that the proportionality between V and I

is a direct consequence of Eq. 7.3: if you want to double V, you simply double the charge on the electrodes—that doubles $\mathbf{E}$, which (for an ohmic material) doubles $\mathbf{J}$, which doubles I.

For *steady* currents and *uniform* conductivity,

$$\nabla \cdot \mathbf{E} = \frac{1}{\sigma}\nabla \cdot \mathbf{J} = 0, \tag{7.5}$$

(Eq. 5.33), and therefore the charge density is zero; any unbalanced charge resides on the surface. (We proved this long ago, for the case of *stationary* charges, using the fact that $\mathbf{E} = \mathbf{0}$; evidently, it is still true when the charges are allowed to move.) It follows, in particular, that Laplace's equation holds within a homogeneous ohmic material carrying a steady current, so all the tools and tricks of Chapter 3 are available for calculating the potential.

Example 7.3. I asserted that the field in Ex. 7.1 is *uniform*. Let's prove it.

Solution

Within the cylinder V obeys Laplace's equation. What are the boundary conditions? At the left end the potential is constant—we may as well set it equal to zero. At the right end the potential is likewise constant—call it V_0. On the cylindrical surface, $\mathbf{J} \cdot \hat{\mathbf{n}} = 0$, or else charge would be leaking out into the surrounding space (which we take to be nonconducting). Therefore $\mathbf{E} \cdot \hat{\mathbf{n}} = 0$, and hence $\partial V/\partial n = 0$. With V or its normal derivative specified on all surfaces, the potential is uniquely determined (Prob. 3.5). But it's easy to guess *one* potential that obeys Laplace's equation and fits these boundary conditions:

$$V(z) = \frac{V_0 z}{L},$$

where z is measured along the axis. The uniqueness theorem guarantees that this is *the* solution. The corresponding field is

$$\mathbf{E} = -\nabla V = -\frac{V_0}{L}\hat{\mathbf{z}},$$

which is indeed uniform. □

Contrast the enormously more difficult problem that arises if the conducting material is removed, leaving only a metal plate at either end (Fig. 7.3). Evidently

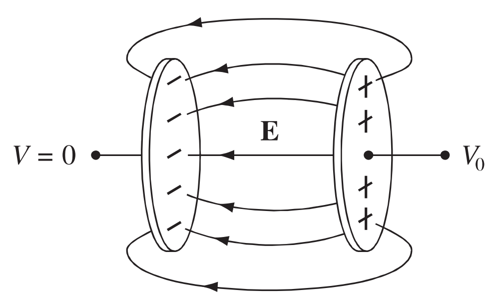


FIGURE 7.3

in the present case charge arranges itself over the surface of the wire in just such a way as to produce a nice uniform field within.[1]

I don't suppose there is any formula in physics more familiar than Ohm's law, and yet it's not really a true law, in the sense of Coulomb's or Ampère's; rather, it is a "rule of thumb" that applies pretty well to many substances. You're not going to win a Nobel prize for finding an exception. In fact, when you stop to think about it, it's a little surprising that Ohm's law *ever* holds. After all, a given field $\mathbf{E}$ produces a force $q\mathbf{E}$ (on a charge q), and according to Newton's second law, the charge will accelerate. But if the charges are *accelerating*, why doesn't the current *increase* with time, growing larger and larger the longer you leave the field on? Ohm's law implies, on the contrary, that a constant field produces a constant *current*, which suggests a constant *velocity*. Isn't that a contradiction to Newton's law?

No, for we are forgetting the frequent collisions electrons make as they pass down the wire. It's a little like this: Suppose you're driving down a street with a stop sign at every intersection, so that, although you accelerate constantly in between, you are obliged to start all over again with each new block. Your *average* speed is then a constant, in spite of the fact that (save for the periodic abrupt stops) you are always accelerating. If the length of a block is λ and your acceleration is a, the time it takes to go a block is

$$t = \sqrt{\frac{2\lambda}{a}},$$

and hence your average velocity is

$$v_{\text{ave}} = \frac{1}{2}at = \sqrt{\frac{\lambda a}{2}}.$$

But wait! That's no good *either!* It says that the velocity is proportional to the *square root* of the acceleration, and therefore that the current should be proportional to the square root of the field! There's another twist to the story: In practice, the charges are already moving very fast because of their thermal energy. But the thermal velocities have random directions, and average to zero. The **drift velocity** we are concerned with is a tiny extra bit (Prob. 5.20). So the time between collisions is actually much shorter than we supposed; if we assume for the sake of argument that all charges travel the same distance λ between collisions, then

$$t = \frac{\lambda}{v_{\text{thermal}}},$$

and therefore

$$v_{\text{ave}} = \frac{1}{2}at = \frac{a\lambda}{2v_{\text{thermal}}}.$$

[1] *Calculating* this surface charge is not easy. See, for example, J. D. Jackson, *Am. J. Phys.* **64**, 855 (1996). Nor is it a simple matter to determine the field *outside* the wire—see Prob. 7.43.

If there are n molecules per unit volume, and f free electrons per molecule, each with charge q and mass m, the current density is

$$\mathbf{J} = nfq\mathbf{v}_{\text{ave}} = \frac{nfq\lambda}{2v_{\text{thermal}}}\frac{\mathbf{F}}{m} = \left(\frac{nf\lambda q^2}{2mv_{\text{thermal}}}\right)\mathbf{E}. \tag{7.6}$$

I don't claim that the term in parentheses is an accurate formula for the conductivity,[2] but it does indicate the basic ingredients, and it correctly predicts that conductivity is proportional to the density of the moving charges and (ordinarily) decreases with increasing temperature.

As a result of all the collisions, the work done by the electrical force is converted into heat in the resistor. Since the work done per unit charge is V and the charge flowing per unit time is I, the power delivered is

$$\boxed{P = VI = I^2R.} \tag{7.7}$$

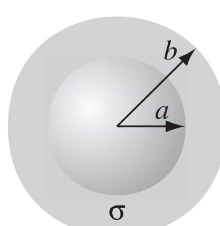
This is the **Joule heating law**. With I in amperes and R in ohms, P comes out in watts (joules per second).

Problem 7.1 Two concentric metal spherical shells, of radius a and b, respectively, are separated by weakly conducting material of conductivity σ (Fig. 7.4a).

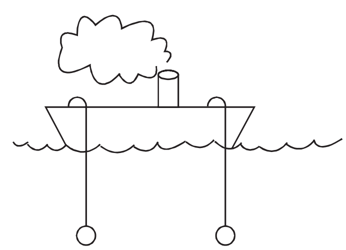
(a) If they are maintained at a potential difference V, what current flows from one to the other?

(b) What is the resistance between the shells?

(c) Notice that if $b \gg a$ the outer radius (b) is irrelevant. How do you account for that? Exploit this observation to determine the current flowing between two metal spheres, each of radius a, immersed deep in the sea and held quite far apart (Fig. 7.4b), if the potential difference between them is V. (This arrangement can be used to measure the conductivity of sea water.)



(a)



(b)

FIGURE 7.4

[2]This classical model (due to Drude) bears little resemblance to the modern quantum theory of conductivity. See, for instance, D. Park's *Introduction to the Quantum Theory*, 3rd ed., Chap. 15 (New York: McGraw-Hill, 1992).

Problem 7.2 A capacitor C has been charged up to potential V_0; at time $t = 0$, it is connected to a resistor R, and begins to discharge (Fig. 7.5a).

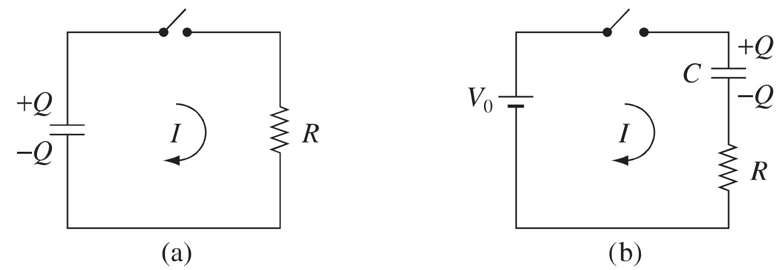


FIGURE 7.5

(a) Determine the charge on the capacitor as a function of time, $Q(t)$. What is the current through the resistor, $I(t)$?

(b) What was the original energy stored in the capacitor (Eq. 2.55)? By integrating Eq. 7.7, confirm that the heat delivered to the resistor is equal to the energy lost by the capacitor.

Now imagine *charging up* the capacitor, by connecting it (and the resistor) to a battery of voltage V_0, at time $t = 0$ (Fig. 7.5b).

(c) Again, determine $Q(t)$ and $I(t)$.

(d) Find the total energy output of the battery ($\int V_0 I \, dt$). Determine the heat delivered to the resistor. What is the final energy stored in the capacitor? What fraction of the work done by the battery shows up as energy in the capacitor? [Notice that the answer is independent of R!]

Problem 7.3

(a) Two metal objects are embedded in weakly conducting material of conductivity σ (Fig. 7.6). Show that the resistance between them is related to the capacitance of the arrangement by

$$R = \frac{\epsilon_0}{\sigma C}.$$

(b) Suppose you connected a battery between 1 and 2, and charged them up to a potential difference V_0. If you then disconnect the battery, the charge will gradually leak off. Show that $V(t) = V_0 e^{-t/\tau}$, and find the **time constant**, τ, in terms of ϵ_0 and σ.

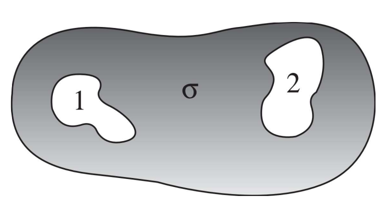


FIGURE 7.6

Problem 7.4 Suppose the conductivity of the material separating the cylinders in Ex. 7.2 is not uniform; specifically, $\sigma(s) = k/s$, for some constant k. Find the resistance between the cylinders. [*Hint:* Because σ is a function of position, Eq. 7.5 does not hold, the charge density is not zero in the resistive medium, and **E** does not go like $1/s$. But we *do* know that for steady currents I is the same across each cylindrical surface. Take it from there.]

7.1.2 ■ Electromotive Force

If you think about a typical electric circuit—a battery hooked up to a light bulb, say (Fig. 7.7)—a perplexing question arises: In practice, the *current is the same all the way around the loop*; why is this the case, when the only obvious driving force is inside the battery? Off hand, you might expect a large current in the battery and none at all in the lamp. Who's doing the pushing, in the rest of the circuit, and how does it happen that this push is exactly right to produce the same current in each segment? What's more, given that the charges in a typical wire move (literally) at a *snail's* pace (see Prob. 5.20), why doesn't it take half an hour for the current to reach the light bulb? How do all the charges know to start moving at the same instant?

Answer: If the current were *not* the same all the way around (for instance, during the first split second after the switch is closed), then charge would be piling up somewhere, and—here's the crucial point—the electric field of this accumulating charge is in such a direction as to even out the flow. Suppose, for instance, that the current *into* the bend in Fig. 7.8 is greater than the current *out*. Then charge piles up at the "knee," and this produces a field aiming *away* from the kink.[3] This field *opposes* the current flowing in (slowing it down) and *promotes* the current flowing out (speeding it up) until these currents are equal, at which point there is no further accumulation of charge, and equilibrium is established. It's a beautiful system, automatically self-correcting to keep the current uniform, and it does it all so quickly that, in practice, you can safely assume the current is the same all around the circuit, even in systems that oscillate at radio frequencies.

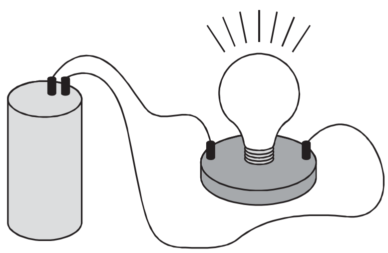


FIGURE 7.7

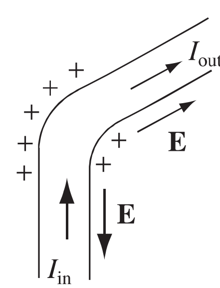


FIGURE 7.8

[3]The amount of charge involved is surprisingly small—see W. G. V. Rosser, *Am. J. Phys.* **38**, 265 (1970); nevertheless, the resulting field can be detected experimentally—see R. Jacobs, A. de Salazar, and A. Nassar, *Am. J. Phys.* **78**, 1432 (2010).

There are really *two* forces involved in driving current around a circuit: the *source*, $\mathbf{f}_s$, which is ordinarily confined to one portion of the loop (a battery, say), and an *electrostatic* force, which serves to smooth out the flow and communicate the influence of the source to distant parts of the circuit:

$$\mathbf{f} = \mathbf{f}_s + \mathbf{E}. \tag{7.8}$$

The physical agency responsible for $\mathbf{f}_s$ can be many different things: in a battery it's a chemical force; in a piezoelectric crystal mechanical pressure is converted into an electrical impulse; in a thermocouple it's a temperature gradient that does the job; in a photoelectric cell it's light; and in a Van de Graaff generator the electrons are literally loaded onto a conveyer belt and swept along. Whatever the *mechanism*, its net effect is determined by the line integral of $\mathbf{f}$ around the circuit:

$$\boxed{\mathcal{E} \equiv \oint \mathbf{f} \cdot d\mathbf{l} = \oint \mathbf{f}_s \cdot d\mathbf{l}.} \tag{7.9}$$

(Because $\oint \mathbf{E} \cdot d\mathbf{l} = 0$ for electrostatic fields, it doesn't matter whether you use $\mathbf{f}$ or $\mathbf{f}_s$.) $\mathcal{E}$ is called the **electromotive force**, or **emf**, of the circuit. It's a lousy term, since this is not a *force* at all—it's the *integral* of a *force per unit charge*. Some people prefer the word **electromotance**, but emf is so established that I think we'd better stick with it.

Within an ideal source of emf (a resistanceless battery,[4] for instance), the *net* force on the charges is *zero* (Eq. 7.1 with $\sigma = \infty$), so $\mathbf{E} = -\mathbf{f}_s$. The potential difference between the terminals (a and b) is therefore

$$V = -\int_a^b \mathbf{E} \cdot d\mathbf{l} = \int_a^b \mathbf{f}_s \cdot d\mathbf{l} = \oint \mathbf{f}_s \cdot d\mathbf{l} = \mathcal{E} \tag{7.10}$$

(we can extend the integral to the entire loop because $\mathbf{f}_s = \mathbf{0}$ outside the source). The function of a battery, then, is to establish and maintain a voltage difference equal to the electromotive force (a 6 V battery, for example, holds the positive terminal 6 V above the negative terminal). The resulting electrostatic field drives current around the rest of the circuit (notice, however, that *inside* the battery $\mathbf{f}_s$ drives current in the direction *opposite* to $\mathbf{E}$).[5]

Because it's the line integral of $\mathbf{f}_s$, $\mathcal{E}$ can be interpreted as the *work done per unit charge*, by the source—indeed, in some books electromotive force is *defined* this way. However, as you'll see in the next section, there is some subtlety involved in this interpretation, so I prefer Eq. 7.9.

[4] *Real* batteries have a certain **internal resistance**, r, and the potential difference between their terminals is $\mathcal{E} - Ir$, when a current I is flowing. For an illuminating discussion of how batteries work, see D. Roberts, *Am. J. Phys.* **51**, 829 (1983).

[5] Current in an electric circuit is somewhat analogous to the flow of water in a closed system of pipes, with gravity playing the role of the electrostatic field, and a pump (lifting the water up *against* gravity) in the role of the battery. In this story *height* is analogous to voltage.

Problem 7.5 A battery of emf $\mathcal{E}$ and internal resistance r is hooked up to a variable "load" resistance, R. If you want to deliver the maximum possible power to the load, what resistance R should you choose? (You can't change $\mathcal{E}$ and r, of course.)

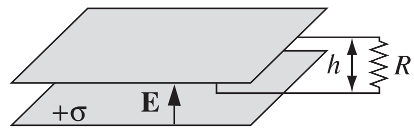


FIGURE 7.9

Problem 7.6 A rectangular loop of wire is situated so that one end (height h) is between the plates of a parallel-plate capacitor (Fig. 7.9), oriented parallel to the field **E**. The other end is way outside, where the field is essentially zero. What is the emf in this loop? If the total resistance is R, what current flows? Explain. [*Warning:* This is a trick question, so be careful; if you have invented a perpetual motion machine, there's probably something wrong with it.]

7.1.3 ■ Motional emf

In the last section, I listed several possible sources of electromotive force, batteries being the most familiar. But I did not mention the commonest one of all: the **generator**. Generators exploit **motional emfs**, which arise when you *move a wire through a magnetic field*. Figure 7.10 suggests a primitive model for a generator. In the shaded region there is a uniform magnetic field **B**, pointing into the page, and the resistor R represents whatever it is (maybe a light bulb or a toaster) we're trying to drive current through. If the entire loop is pulled to the right with speed v, the charges in segment ab experience a magnetic force whose vertical component qvB drives current around the loop, in the clockwise direction. The emf is

$$\mathcal{E} = \oint \mathbf{f}_{\text{mag}} \cdot d\mathbf{l} = vBh, \tag{7.11}$$

where h is the width of the loop. (The horizontal segments bc and ad contribute nothing, since the force there is perpendicular to the wire.)

Notice that the integral you perform to calculate $\mathcal{E}$ (Eq. 7.9 or 7.11) is carried out at *one instant of time*—take a "snapshot" of the loop, if you like, and work

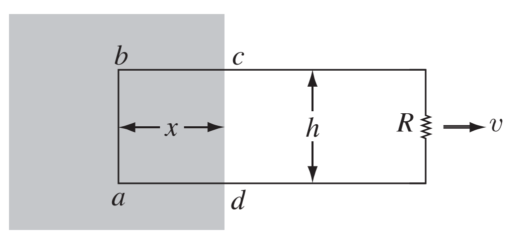


FIGURE 7.10

from that. Thus $d\mathbf{l}$, for the segment ab in Fig. 7.10, points straight up, even though the loop is moving to the right. You can't quarrel with this—it's simply the way emf is *defined*—but it is important to be clear about it.

In particular, although the magnetic force is responsible for establishing the emf, it is *not* doing any work—magnetic forces *never* do work. Who, then, *is* supplying the energy that heats the resistor? *Answer:* The person who's pulling on the loop. With the current flowing, the free charges in segment ab have a vertical velocity (call it $\mathbf{u}$) in addition to the horizontal velocity $\mathbf{v}$ they inherit from the motion of the loop. Accordingly, the magnetic force has a component quB to the left. To counteract this, the person pulling on the wire must exert a force per unit charge

$$f_{\text{pull}} = uB$$

to the *right* (Fig. 7.11). This force is transmitted to the charge by the structure of the wire.

Meanwhile, the particle is actually *moving* in the direction of the resultant velocity $\mathbf{w}$, and the distance it goes is $(h/\cos\theta)$. The work done per unit charge is therefore

$$\int \mathbf{f}_{\text{pull}} \cdot d\mathbf{l} = (uB)\left(\frac{h}{\cos\theta}\right)\sin\theta = vBh = \mathcal{E}$$

($\sin\theta$ coming from the dot product). As it turns out, then, the *work done per unit charge is exactly equal to the emf*, though the integrals are taken along entirely different paths (Fig. 7.12), and completely different forces are involved. To calculate the emf, you integrate around the loop at *one instant*, but to calculate the work done you follow a charge in its journey around the loop; $\mathbf{f}_{\text{pull}}$ contributes nothing to the emf, because it is perpendicular to the wire, whereas $\mathbf{f}_{\text{mag}}$ contributes nothing to work because it is perpendicular to the motion of the charge.[6]

There is a particularly nice way of expressing the emf generated in a moving loop. Let Φ be the flux of $\mathbf{B}$ through the loop:

$$\Phi \equiv \int \mathbf{B} \cdot d\mathbf{a}. \tag{7.12}$$

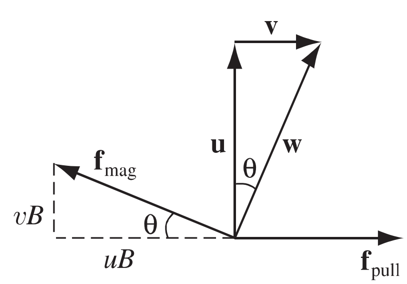
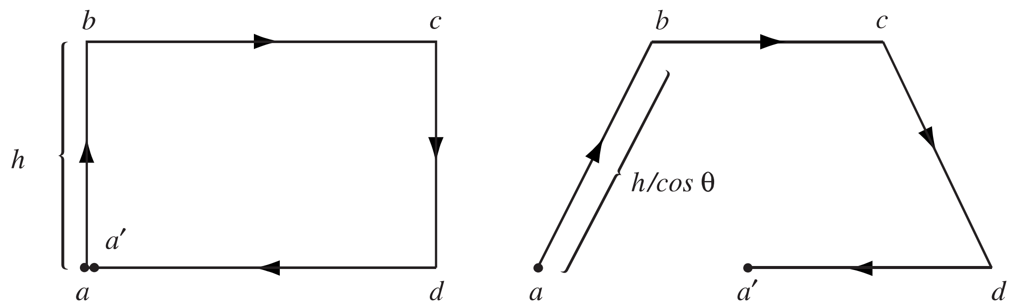


FIGURE 7.11

[6]For further discussion, see E. P. Mosca, *Am. J. Phys.* **42**, 295 (1974).



(a) Integration path for computing $\mathcal{E}$ (follow the wire at one instant of time).

(b) Integration path for calculating work done (follow the charge around the loop).

FIGURE 7.12

For the rectangular loop in Fig. 7.10,

$$\Phi = Bhx.$$

As the loop moves, the flux decreases:

$$\frac{d\Phi}{dt} = Bh\frac{dx}{dt} = -Bhv.$$

(The minus sign accounts for the fact that dx/dt is negative.) But this is precisely the emf (Eq. 7.11); evidently the emf generated in the loop is minus the rate of change of flux through the loop:

$$\boxed{\mathcal{E} = -\frac{d\Phi}{dt}.} \tag{7.13}$$

This is the **flux rule** for motional emf.

Apart from its delightful simplicity, the flux rule has the virtue of applying to *non*rectangular loops moving in *arbitrary* directions through *non*uniform magnetic fields; in fact, the loop need not even maintain a fixed shape.

Proof. Figure 7.13 shows a loop of wire at time t, and also a short time dt later. Suppose we compute the flux at time t, using surface $\mathcal{S}$, and the flux at time $t + dt$, using the surface consisting of $\mathcal{S}$ plus the "ribbon" that connects the new position of the loop to the old. The *change* in flux, then, is

$$d\Phi = \Phi(t + dt) - \Phi(t) = \Phi_{\text{ribbon}} = \int_{\text{ribbon}} \mathbf{B} \cdot d\mathbf{a}.$$

Focus your attention on point P: in time dt, it moves to P'. Let $\mathbf{v}$ be the velocity of the *wire*, and $\mathbf{u}$ the velocity of a charge *down* the wire; $\mathbf{w} = \mathbf{v} + \mathbf{u}$ is the resultant

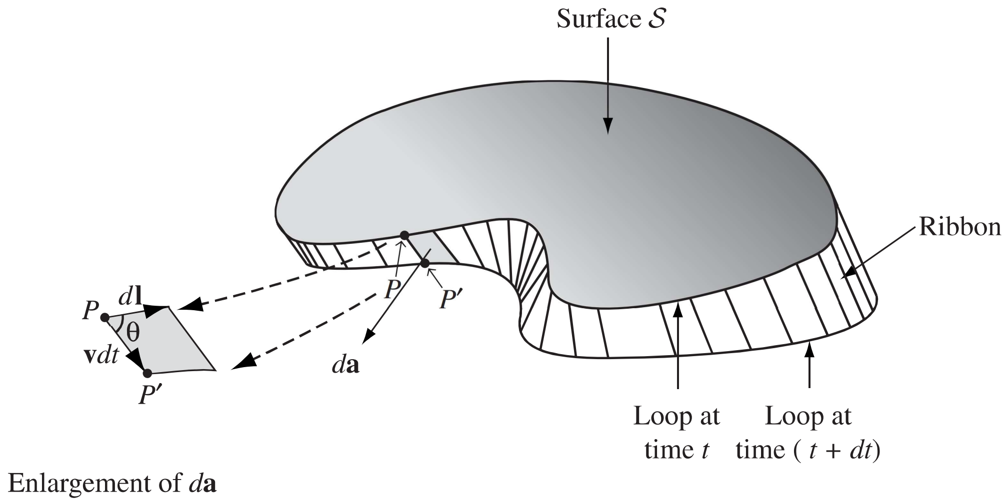


FIGURE 7.13

velocity of a charge at P. The infinitesimal element of area on the ribbon can be written as

$$d\mathbf{a} = (\mathbf{v} \times d\mathbf{l})\, dt$$

(see inset in Fig. 7.13). Therefore

$$\frac{d\Phi}{dt} = \oint \mathbf{B} \cdot (\mathbf{v} \times d\mathbf{l}).$$

Since $\mathbf{w} = (\mathbf{v} + \mathbf{u})$ and $\mathbf{u}$ is parallel to $d\mathbf{l}$, we can just as well write this as

$$\frac{d\Phi}{dt} = \oint \mathbf{B} \cdot (\mathbf{w} \times d\mathbf{l}).$$

Now, the scalar triple-product can be rewritten:

$$\mathbf{B} \cdot (\mathbf{w} \times d\mathbf{l}) = -(\mathbf{w} \times \mathbf{B}) \cdot d\mathbf{l},$$

so

$$\frac{d\Phi}{dt} = -\oint (\mathbf{w} \times \mathbf{B}) \cdot d\mathbf{l}.$$

But $(\mathbf{w} \times \mathbf{B})$ is the magnetic force per unit charge, $\mathbf{f}_{\text{mag}}$, so

$$\frac{d\Phi}{dt} = -\oint \mathbf{f}_{\text{mag}} \cdot d\mathbf{l},$$

and the integral of $\mathbf{f}_{\text{mag}}$ is the emf:

$$\mathcal{E} = -\frac{d\Phi}{dt}. \qquad \square$$

There is a sign ambiguity in the definition of emf (Eq. 7.9): Which *way* around the loop are you supposed to integrate? There is a compensatory ambiguity in the definition of *flux* (Eq. 7.12): Which is the positive direction for $d\mathbf{a}$? In applying

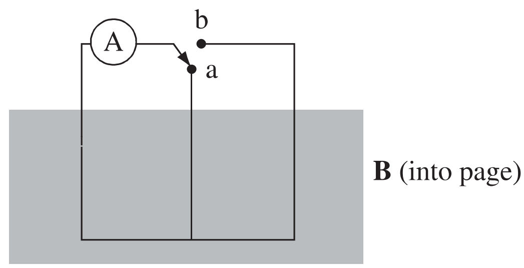


FIGURE 7.14

the flux rule, sign consistency is governed (as always) by your right hand: If your fingers define the positive direction around the loop, then your thumb indicates the direction of $d\mathbf{a}$. Should the emf come out negative, it means the current will flow in the negative direction around the circuit.

The flux rule is a nifty short-cut for calculating motional emfs. It does not contain any new physics—just the Lorentz force law. But it can lead to error or ambiguity if you're not careful. The flux rule assumes you have a single wire loop—it can move, rotate, stretch, or distort (continuously), but beware of switches, sliding contacts, or extended conductors allowing a variety of current paths. A standard "flux rule paradox" involves the circuit in Figure 7.14. When the switch is thrown (from a to b) the flux through the circuit doubles, but there's no motional emf (no conductor moving through a magnetic field), and the ammeter (A) records no current.

Example 7.4. A metal disk of radius a rotates with angular velocity ω about a vertical axis, through a uniform field $\mathbf{B}$, pointing up. A circuit is made by connecting one end of a resistor to the axle and the other end to a sliding contact, which touches the outer edge of the disk (Fig. 7.15). Find the current in the resistor.

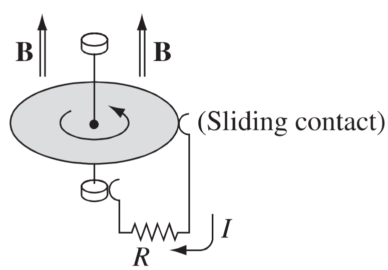


FIGURE 7.15

Solution

The speed of a point on the disk at a distance s from the axis is $v = \omega s$, so the force per unit charge is $\mathbf{f}_{\text{mag}} = \mathbf{v} \times \mathbf{B} = \omega s B \hat{\mathbf{s}}$. The emf is therefore

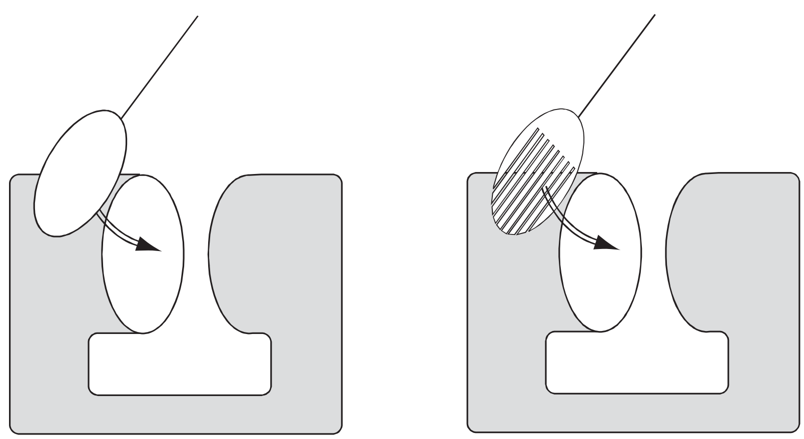
$$\mathcal{E} = \int_0^a f_{\text{mag}}\, ds = \omega B \int_0^a s\, ds = \frac{\omega B a^2}{2},$$

and the current is

$$I = \frac{\mathcal{E}}{R} = \frac{\omega B a^2}{2R}.$$

Example 7.4 (the **Faraday disk**, or **Faraday dynamo**) involves a motional emf that you can't calculate (at least, not directly) from the flux rule. The flux rule assumes the current flows along a well-defined path, whereas in this example the current spreads out over the whole disk. It's not even clear what the "flux through the circuit" would *mean* in this context.

Even more tricky is the case of **eddy currents**. Take a chunk of aluminum (say), and shake it around in a nonuniform magnetic field. Currents will be generated in the material, and you will feel a kind of "viscous drag"—as though you were pulling the block through molasses (this is the force I called $\mathbf{f}_{\text{pull}}$ in the discussion of motional emf). Eddy currents are notoriously difficult to calculate,[7] but easy and dramatic to demonstrate. You may have witnessed the classic experiment in which an aluminum disk mounted as a pendulum on a horizontal axis swings down and passes between the poles of a magnet (Fig. 7.16a). When it enters the field region it suddenly slows way down. To confirm that eddy currents are responsible, one repeats the demonstration using a disk that has many slots cut in it, to prevent the flow of large-scale currents (Fig. 7.16b). This time the disk swings freely, unimpeded by the field.



(a) (b)

FIGURE 7.16

Problem 7.7 A metal bar of mass m slides frictionlessly on two parallel conducting rails a distance l apart (Fig. 7.17). A resistor R is connected across the rails, and a uniform magnetic field $\mathbf{B}$, pointing into the page, fills the entire region.

[7]See, for example, W. M. Saslow, *Am. J. Phys.*, **60**, 693 (1992).

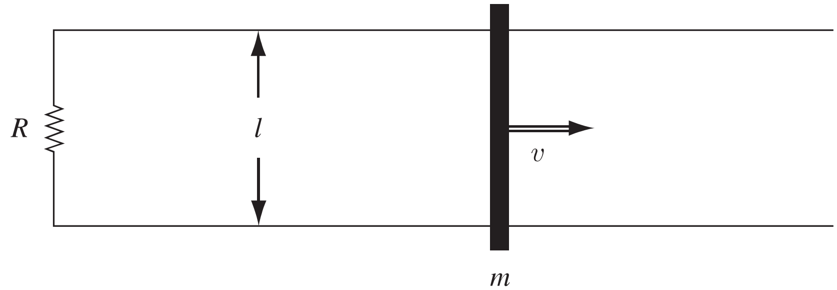


FIGURE 7.17

(a) If the bar moves to the right at speed v, what is the current in the resistor? In what direction does it flow?

(b) What is the magnetic force on the bar? In what direction?

(c) If the bar starts out with speed v_0 at time $t = 0$, and is left to slide, what is its speed at a later time t?

(d) The initial kinetic energy of the bar was, of course, $\frac{1}{2}mv_0{}^2$. Check that the energy delivered to the resistor is exactly $\frac{1}{2}mv_0{}^2$.

Problem 7.8 A square loop of wire (side a) lies on a table, a distance s from a very long straight wire, which carries a current I, as shown in Fig. 7.18.

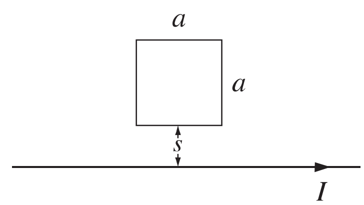


FIGURE 7.18

(a) Find the flux of **B** through the loop.

(b) If someone now pulls the loop directly away from the wire, at speed v, what emf is generated? In what direction (clockwise or counterclockwise) does the current flow?

(c) What if the loop is pulled to the *right* at speed v?

Problem 7.9 An infinite number of different surfaces can be fit to a given boundary line, and yet, in defining the magnetic flux through a loop, $\Phi = \int \mathbf{B} \cdot d\mathbf{a}$, I never specified the particular surface to be used. Justify this apparent oversight.

Problem 7.10 A square loop (side a) is mounted on a vertical shaft and rotated at angular velocity ω (Fig. 7.19). A uniform magnetic field **B** points to the right. Find the $\mathcal{E}(t)$ for this **alternating current** generator.

Problem 7.11 A square loop is cut out of a thick sheet of aluminum. It is then placed so that the top portion is in a uniform magnetic field **B**, and is allowed to fall under gravity (Fig. 7.20). (In the diagram, shading indicates the field region; **B** points into the page.) If the magnetic field is 1 T (a pretty standard laboratory field), find the terminal velocity of the loop (in m/s). Find the velocity of the loop as a function of time. How long does it take (in seconds) to reach, say, 90% of the terminal velocity? What would happen if you cut a tiny slit in the ring, breaking the circuit? [*Note:* The dimensions of the loop cancel out; determine the actual *numbers*, in the units indicated.]

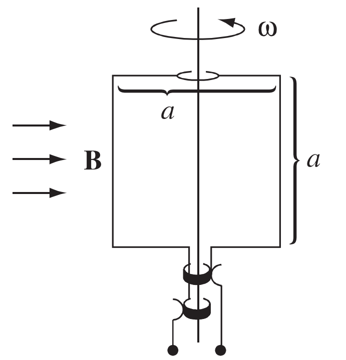


FIGURE 7.19

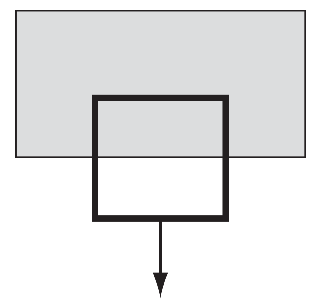


FIGURE 7.20

7.2 ■ ELECTROMAGNETIC INDUCTION

7.2.1 ■ Faraday's Law

In 1831 Michael Faraday reported on a series of experiments, including three that (with some violence to history) can be characterized as follows:

Experiment 1. He pulled a loop of wire to the right through a magnetic field (Fig. 7.21a). A current flowed in the loop.

Experiment 2. He moved the *magnet* to the *left*, holding the loop still (Fig. 7.21b). Again, a current flowed in the loop.

Experiment 3. With both the loop and the magnet at rest (Fig. 7.21c), he changed the *strength* of the field (he used an electromagnet, and varied the current in the coil). Once again, current flowed in the loop.

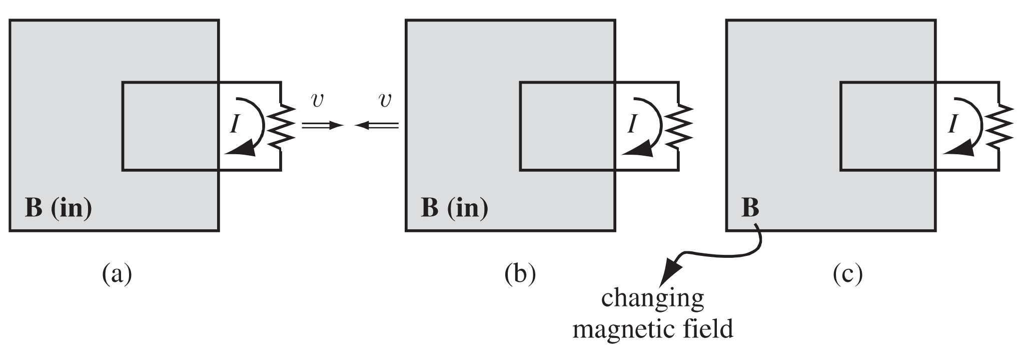


FIGURE 7.21

The first experiment, of course, is a straightforward case of motional emf; according to the flux rule:

$$\mathcal{E} = -\frac{d\Phi}{dt}.$$

I don't think it will surprise you to learn that exactly the same emf arises in Experiment 2—all that really matters is the *relative* motion of the magnet and the loop. Indeed, in the light of special relativity it *has* to be so. But Faraday knew nothing of relativity, and in classical electrodynamics this simple reciprocity is a remarkable coincidence. For if the *loop* moves, it's a *magnetic* force that sets up the emf, but if the loop is *stationary*, the force *cannot* be magnetic—stationary charges experience no magnetic forces. In that case, what *is* responsible? What sort of field exerts a force on charges at rest? Well, *electric* fields do, of course, but in this case there doesn't seem to be any electric field in sight.

Faraday had an ingenious inspiration:

A changing magnetic field induces an electric field.

It is this induced[8] electric field that accounts for the emf in Experiment 2.[9] Indeed, if (as Faraday found empirically) the emf is again equal to the rate of change of the flux,

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}, \tag{7.14}$$

then **E** is related to the change in **B** by the equation

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}. \tag{7.15}$$

This is **Faraday's law**, in integral form. We can convert it to differential form by applying Stokes' theorem:

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.} \tag{7.16}$$

[8]"Induce" is a subtle and slippery verb. It carries a faint odor of *causation* ("*pro*duce" would make this explicit) without quite committing itself. There is a sterile ongoing debate in the literature as to whether a changing magnetic field should be regarded as an independent "source" of electric fields (along with electric charge)—after all, the magnetic field *itself* is due to electric currents. It's like asking whether the postman is the "source" of my mail. Well, sure—he delivered it to my door. On the other hand, Grandma wrote the letter. Ultimately, ρ and **J** are the sources of *all* electromagnetic fields, and a changing magnetic field merely delivers electromagnetic news from currents elsewhere. But it is often convenient to think of a changing magnetic field "producing" an electric field, and it won't hurt you as long as you understand that this is the condensed version of a more complicated story. For a nice discussion, see S. E. Hill, *Phys. Teach.* **48**, 410 (2010).

[9]You might argue that the magnetic field in Experiment 2 is not really *changing*—just *moving*. What I mean is that if you sit at a *fixed location*, the field you experience changes as the magnet passes by.

Note that Faraday's law reduces to the old rule $\oint \mathbf{E} \cdot d\mathbf{l} = 0$ (or, in differential form, $\nabla \times \mathbf{E} = \mathbf{0}$) in the static case (constant $\mathbf{B}$) as, of course, it should.

In Experiment 3, the magnetic field changes for entirely different reasons, but according to Faraday's law an electric field will again be induced, giving rise to an emf $-d\Phi/dt$. Indeed, one can subsume all three cases (and for that matter any combination of them) into a kind of **universal flux rule**:

> **Whenever (and for whatever reason) the magnetic flux through a loop changes, an emf**
>
> $$\mathcal{E} = -\frac{d\Phi}{dt} \tag{7.17}$$
>
> **will appear in the loop**.

Many people call *this* "Faraday's law." Maybe I'm overly fastidious, but I find this confusing. There are really *two* totally different mechanisms underlying Eq. 7.17, and to identify them both as "Faraday's law" is a little like saying that because identical twins look alike we ought to call them by the same name. In Faraday's first experiment it's the Lorentz force law at work; the emf is *magnetic*. But in the other two it's an *electric* field (induced by the changing magnetic field) that does the job. Viewed in this light, it is quite astonishing that all three processes yield the same formula for the emf. In fact, it was precisely this "coincidence" that led Einstein to the special theory of relativity—he sought a deeper understanding of what is, in classical electrodynamics, a peculiar accident. But that's a story for Chapter 12. In the meantime, I shall reserve the term "Faraday's law" for electric fields induced by changing magnetic fields, and I do *not* regard Experiment 1 as an instance of Faraday's law.

Example 7.5. A long cylindrical magnet of length L and radius a carries a uniform magnetization $\mathbf{M}$ parallel to its axis. It passes at constant velocity v through a circular wire ring of slightly larger diameter (Fig. 7.22). Graph the emf induced in the ring, as a function of time.

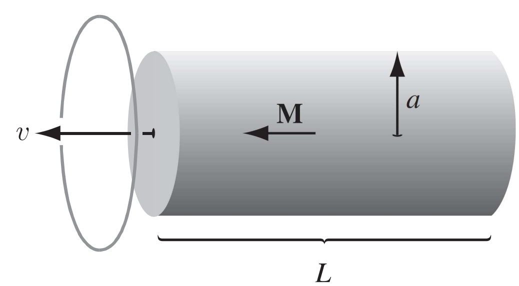


FIGURE 7.22

Solution

The magnetic field is the same as that of a long solenoid with surface current $\mathbf{K}_b = M\,\hat{\boldsymbol{\phi}}$. So the field inside is $\mathbf{B} = \mu_0\mathbf{M}$, except near the ends, where it starts to spread out. The flux through the ring is zero when the magnet is far away; it builds up to a maximum of $\mu_0 M \pi a^2$ as the leading end passes through; and it drops back to zero as the trailing end emerges (Fig. 7.23a). The emf is (minus) the derivative of Φ with respect to time, so it consists of two spikes, as shown in Fig. 7.23b.

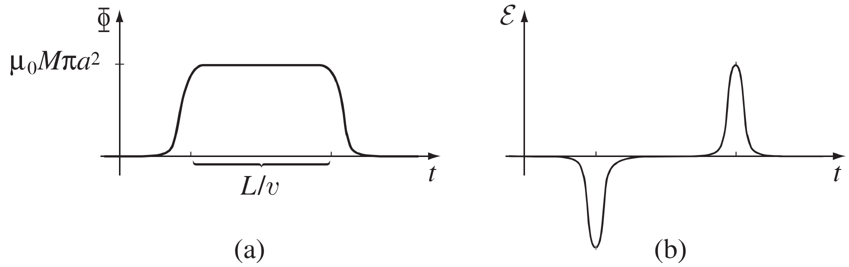


FIGURE 7.23

Keeping track of the *signs* in Faraday's law can be a real headache. For instance, in Ex. 7.5 we would like to know which *way* around the ring the induced current flows. In principle, the right-hand rule does the job (we called Φ positive to the left, in Fig. 7.22, so the positive direction for current in the ring is counterclockwise, as viewed from the left; since the first spike in Fig. 7.23b is *negative*, the first current pulse flows *clockwise*, and the second counterclockwise). But there's a handy rule, called **Lenz's law**, whose sole purpose is to help you get the directions right:[10]

Nature abhors a change in flux.

The induced current will flow in such a direction that the flux *it* produces tends to cancel the change. (As the front end of the magnet in Ex. 7.5 enters the ring, the flux increases, so the current in the ring must generate a field to the *right*—it therefore flows *clockwise*.) Notice that it is the *change* in flux, not the flux itself, that nature abhors (when the tail end of the magnet exits the ring, the flux *drops*, so the induced current flows *counterclockwise*, in an effort to restore it). Faraday induction is a kind of "inertial" phenomenon: A conducting loop "likes" to maintain a constant flux through it; if you try to *change* the flux, the loop responds by sending a current around in such a direction as to frustrate your efforts. (It doesn't *succeed* completely; the flux produced by the induced current is typically only a tiny fraction of the original. All Lenz's law tells you is the *direction* of the flow.)

[10]Lenz's law applies to *motional* emfs, too, but for them it is usually easier to get the direction of the current from the Lorentz force law.

Example 7.6. The "jumping ring" demonstration. If you wind a solenoidal coil around an iron core (the iron is there to beef up the magnetic field), place a metal ring on top, and plug it in, the ring will jump several feet in the air (Fig. 7.24). Why?

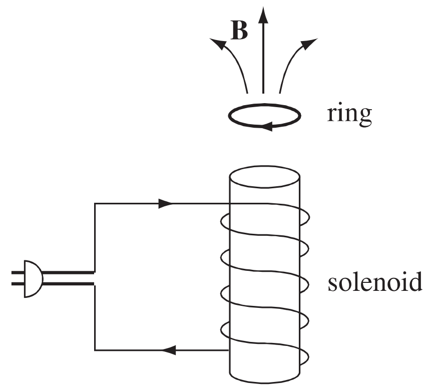


FIGURE 7.24

Solution

Before you turned on the current, the flux through the ring was *zero*. *Afterward* a flux appeared (upward, in the diagram), and the emf generated in the ring led to a current (in the ring) which, according to Lenz's law, was in such a direction that *its* field tended to cancel this new flux. This means that the current in the loop is *opposite* to the current in the solenoid. And opposite currents repel, so the ring flies off.[11]

Problem 7.12 A long solenoid, of radius a, is driven by an alternating current, so that the field inside is sinusoidal: $\mathbf{B}(t) = B_0 \cos(\omega t)\,\hat{\mathbf{z}}$. A circular loop of wire, of radius $a/2$ and resistance R, is placed inside the solenoid, and coaxial with it. Find the current induced in the loop, as a function of time.

Problem 7.13 A square loop of wire, with sides of length a, lies in the first quadrant of the xy plane, with one corner at the origin. In this region, there is a nonuniform time-dependent magnetic field $\mathbf{B}(y, t) = ky^3t^2\,\hat{\mathbf{z}}$ (where k is a constant). Find the emf induced in the loop.

Problem 7.14 As a lecture demonstration a short cylindrical bar magnet is dropped down a vertical aluminum pipe of slightly larger diameter, about 2 meters long. It takes several seconds to emerge at the bottom, whereas an otherwise identical piece of *unmagnetized* iron makes the trip in a fraction of a second. Explain why the magnet falls more slowly.[12]

[11]For further discussion of the jumping ring (and the related "floating ring"), see C. S. Schneider and J. P. Ertel, *Am. J. Phys.* **66**, 686 (1998); P. J. H. Tjossem and E. C. Brost, *Am. J. Phys.* **79**, 353 (2011).
[12]For a discussion of this amazing demonstration see K. D. Hahn et al., *Am. J. Phys.* **66**, 1066 (1998) and G. Donoso, C. L. Ladera, and P. Martin, *Am. J. Phys.* **79**, 193 (2011).

7.2.2 ■ The Induced Electric Field

Faraday's law generalizes the electrostatic rule $\nabla \times \mathbf{E} = \mathbf{0}$ to the time-dependent régime. The *divergence* of $\mathbf{E}$ is still given by Gauss's law ($\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$). If $\mathbf{E}$ is a *pure* Faraday field (due exclusively to a changing $\mathbf{B}$, with $\rho = 0$), then

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

This is mathematically identical to magnetostatics,

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

Conclusion: Faraday-induced electric fields are determined by $-(\partial \mathbf{B}/\partial t)$ in exactly the same way as magnetostatic fields are determined by $\mu_0 \mathbf{J}$. The analog to Biot-Savart is[13] is

$$\mathbf{E} = -\frac{1}{4\pi}\int \frac{(\partial \mathbf{B}/\partial t) \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau = -\frac{1}{4\pi}\frac{\partial}{\partial t}\int \frac{\mathbf{B} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau, \tag{7.18}$$

and if symmetry permits, we can use all the tricks associated with Ampère's law in integral form ($\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$), only now it's *Faraday's* law in integral form:

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}. \tag{7.19}$$

The rate of change of (magnetic) flux through the Amperian loop plays the role formerly assigned to $\mu_0 I_{\text{enc}}$.

Example 7.7. A uniform magnetic field $\mathbf{B}(t)$, pointing straight up, fills the shaded circular region of Fig. 7.25. If $\mathbf{B}$ is changing with time, what is the induced electric field?

Solution

$\mathbf{E}$ points in the circumferential direction, just like the *magnetic* field inside a long straight wire carrying a uniform *current* density. Draw an Amperian loop of radius s, and apply Faraday's law:

$$\oint \mathbf{E} \cdot d\mathbf{l} = E(2\pi s) = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left(\pi s^2 B(t)\right) = -\pi s^2 \frac{dB}{dt}.$$

Therefore

$$\mathbf{E} = -\frac{s}{2}\frac{dB}{dt}\,\hat{\boldsymbol{\phi}}.$$

If $\mathbf{B}$ is *increasing*, $\mathbf{E}$ runs *clockwise*, as viewed from above.

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[13] Magnetostatics holds only for time-independent currents, but there is no such restriction on $\partial \mathbf{B}/\partial t$.

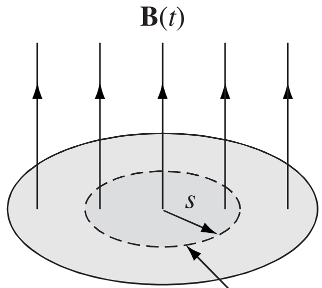

FIGURE 7.25

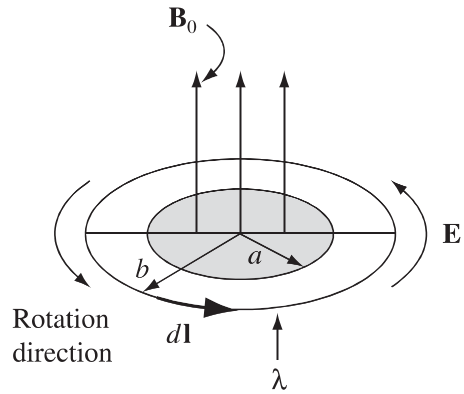

FIGURE 7.26

Example 7.8. A line charge λ is glued onto the rim of a wheel of radius b, which is then suspended horizontally, as shown in Fig. 7.26, so that it is free to rotate (the spokes are made of some nonconducting material—wood, maybe). In the central region, out to radius a, there is a uniform magnetic field $\mathbf{B}_0$, pointing up. Now someone turns the field off. What happens?

Solution

The changing magnetic field will induce an electric field, curling around the axis of the wheel. This electric field exerts a force on the charges at the rim, and the wheel starts to turn. According to Lenz's law, it will rotate in such a direction that *its* field tends to restore the upward flux. The motion, then, is counterclockwise, as viewed from above.

Faraday's law, applied to the loop at radius b, says

$$\oint \mathbf{E} \cdot d\mathbf{l} = E(2\pi b) = -\frac{d\Phi}{dt} = -\pi a^2 \frac{dB}{dt}, \quad \text{or} \quad \mathbf{E} = -\frac{a^2}{2b}\frac{dB}{dt}\,\hat{\boldsymbol{\phi}}.$$

The torque on a segment of length $d\mathbf{l}$ is $(\mathbf{r} \times \mathbf{F})$, or $b\lambda E\,dl$. The total torque on the wheel is therefore

$$N = b\lambda\left(-\frac{a^2}{2b}\frac{dB}{dt}\right)\oint dl = -b\lambda\pi a^2 \frac{dB}{dt},$$

and the angular momentum imparted to the wheel is

$$\int N\,dt = -\lambda\pi a^2 b \int_{B_0}^{0} dB = \lambda\pi a^2 b B_0.$$

It doesn't matter how quickly or slowly you turn off the field; the resulting angular velocity of the wheel is the same regardless. (If you find yourself wondering where the angular momentum *came* from, you're getting ahead of the story! Wait for the next chapter.)

Note that it's the *electric* field that did the rotating. To convince you of this, I deliberately set things up so that the *magnetic* field is *zero* at the location of the charge. The experimenter may tell you she never put in any electric field—all she did was switch off the magnetic field. But when she did that, an electric field automatically appeared, and it's this electric field that turned the wheel.

I must warn you, now, of a small fraud that tarnishes many applications of Faraday's law: Electromagnetic induction, of course, occurs only when the magnetic fields are *changing*, and yet we would like to use the apparatus of magneto*statics* (Ampère's law, the Biot-Savart law, and the rest) to *calculate* those magnetic fields. Technically, any result derived in this way is only approximately correct. But in practice the error is usually negligible, unless the field fluctuates extremely rapidly, or you are interested in points very far from the source. Even the case of a wire snipped by a pair of scissors (Prob. 7.18) is *static enough* for Ampère's law to apply. This régime, in which magnetostatic rules can be used to calculate the magnetic field on the right hand side of Faraday's law, is called **quasistatic**. Generally speaking, it is only when we come to electromagnetic waves and radiation that we must worry seriously about the breakdown of magnetostatics itself.

Example 7.9. An infinitely long straight wire carries a slowly varying current $I(t)$. Determine the induced electric field, as a function of the distance s from the wire.[14]

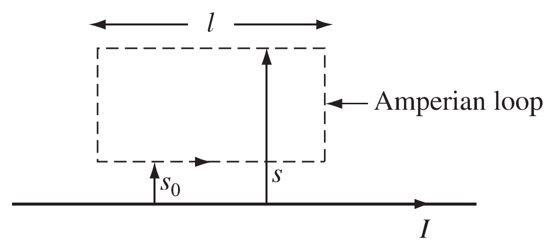

FIGURE 7.27

Solution

In the quasistatic approximation, the magnetic field is $(\mu_0 I/2\pi s)$, and it circles around the wire. Like the **B**-field of a solenoid, **E** here runs parallel to the axis. For the rectangular "Amperian loop" in Fig. 7.27, Faraday's law gives:

$$\begin{aligned}\oint \mathbf{E}\cdot d\mathbf{l} &= E(s_0)l - E(s)l = -\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{a} \\ &= -\frac{\mu_0 l}{2\pi}\frac{dI}{dt}\int_{s_0}^{s}\frac{1}{s'}\,ds' = -\frac{\mu_0 l}{2\pi}\frac{dI}{dt}(\ln s - \ln s_0).\end{aligned}$$

[14]This example is artificial, and not just in the obvious sense of involving infinite wires, but in a more subtle respect. It assumes that the current is the same (at any given instant) all the way down the line. This is a safe assumption for the *short* wires in typical electric circuits, but not for *long* wires (**transmission lines**), unless you supply a distributed and synchronized driving mechanism. But never mind—the problem doesn't inquire how you would *produce* such a current; it only asks what *fields* would result if you *did*. Variations on this problem are discussed by M. A. Heald, *Am. J. Phys.* **54**, 1142 (1986).

Thus

$$\mathbf{E}(s) = \left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln s + K\right]\hat{\mathbf{z}}, \tag{7.20}$$

where K is a constant (that is to say, it is independent of s—it might still be a function of t). The actual *value* of K depends on the whole history of the function $I(t)$—we'll see some examples in Chapter 10.

Equation 7.20 has the peculiar implication that E blows up as s goes to infinity. *That* can't be true ... What's gone wrong? *Answer:* We have overstepped the limits of the quasistatic approximation. As we shall see in Chapter 9, electromagnetic "news" travels at the speed of light, and at large distances **B** depends not on the current *now*, but on the current *as it was* at some earlier time (indeed, a whole *range* of earlier times, since different points on the wire are different distances away). If τ is the time it takes I to change substantially, then the quasistatic approximation should hold only for

$$s \ll c\tau, \tag{7.21}$$

and hence Eq. 7.20 simply does not apply, at extremely large s.

Problem 7.15 A long solenoid with radius a and n turns per unit length carries a time-dependent current $I(t)$ in the $\hat{\boldsymbol{\phi}}$ direction. Find the electric field (magnitude and direction) at a distance s from the axis (both inside and outside the solenoid), in the quasistatic approximation.

Problem 7.16 An alternating current $I = I_0 \cos(\omega t)$ flows down a long straight wire, and returns along a coaxial conducting tube of radius a.

(a) In what *direction* does the induced electric field point (radial, circumferential, or longitudinal)?

(b) Assuming that the field goes to zero as $s \to \infty$, find $\mathbf{E}(s, t)$.[15]

Problem 7.17 A long solenoid of radius a, carrying n turns per unit length, is looped by a wire with resistance R, as shown in Fig. 7.28.

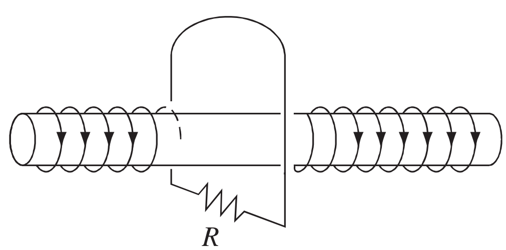

FIGURE 7.28

[15]This is not at all the way electric fields *actually* behave in coaxial cables, for reasons suggested in the previous footnote. See Sect. 9.5.3, or J. G. Cherveniak, *Am. J. Phys.*, **54**, 946 (1986), for a more realistic treatment.

(a) If the current in the solenoid is increasing at a constant rate ($dI/dt = k$), what current flows in the loop, and which way (left or right) does it pass through the resistor?

(b) If the current I in the solenoid is constant but the solenoid is pulled out of the loop (toward the left, to a place far from the loop), what total charge passes through the resistor?

Problem 7.18 A square loop, side a, resistance R, lies a distance s from an infinite straight wire that carries current I (Fig. 7.29). Now someone cuts the wire, so I drops to zero. In what direction does the induced current in the square loop flow, and what total charge passes a given point in the loop during the time this current flows? If you don't like the scissors model, turn the current down *gradually*:

$$I(t) = \begin{cases} (1 - \alpha t)I, & \text{for } 0 \le t \le 1/\alpha, \\ 0, & \text{for } t > 1/\alpha. \end{cases}$$

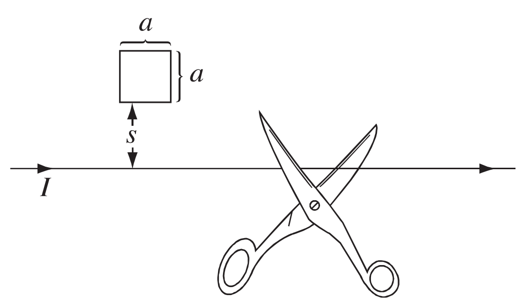

FIGURE 7.29

Problem 7.19 A toroidal coil has a rectangular cross section, with inner radius a, outer radius $a + w$, and height h. It carries a total of N tightly wound turns, and the current is increasing at a constant rate ($dI/dt = k$). If w and h are both much less than a, find the electric field at a point z above the center of the toroid. [*Hint:* Exploit the analogy between Faraday fields and magnetostatic fields, and refer to Ex. 5.6.]

Problem 7.20 Where is $\partial \mathbf{B}/\partial t$ nonzero, in Figure 7.21(b)? Exploit the analogy between Faraday's law and Ampère's law to sketch (qualitatively) the electric field.

Problem 7.21 Imagine a uniform magnetic field, pointing in the z direction and filling all space ($\mathbf{B} = B_0\,\hat{\mathbf{z}}$). A positive charge is at rest, at the origin. Now somebody turns off the magnetic field, thereby inducing an electric field. In what direction does the charge move?[16]

7.2.3 ■ Inductance

Suppose you have two loops of wire, at rest (Fig. 7.30). If you run a steady current I_1 around loop 1, it produces a magnetic field $\mathbf{B}_1$. Some of the field lines pass

[16]This paradox was suggested by Tom Colbert. Refer to Problem 2.55.

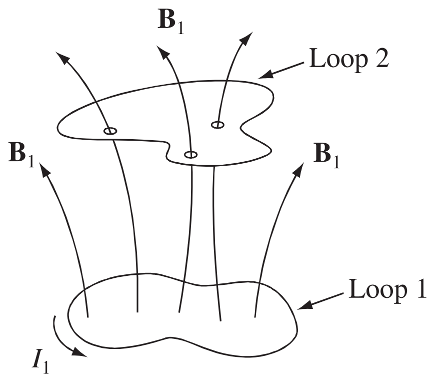

FIGURE 7.30

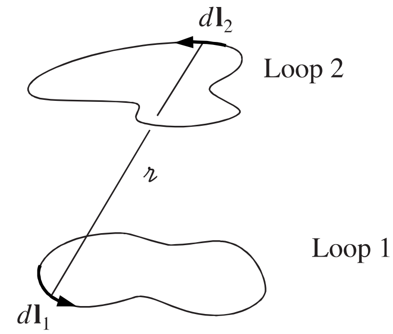

FIGURE 7.31

through loop 2; let Φ_2 be the flux of $\mathbf{B}_1$ through 2. You might have a tough time actually *calculating* $\mathbf{B}_1$, but a glance at the Biot-Savart law,

$$\mathbf{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\mathbf{l}_1 \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2},$$

reveals one significant fact about this field: *It is proportional to the current* I_1. Therefore, so too is the flux through loop 2:

$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}_2.$$

Thus

$$\Phi_2 = M_{21} I_1, \tag{7.22}$$

where M_{21} is the constant of proportionality; it is known as the **mutual inductance** of the two loops.

There is a cute formula for the mutual inductance, which you can derive by expressing the flux in terms of the vector potential, and invoking Stokes' theorem:

$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}_2 = \int (\boldsymbol{\nabla} \times \mathbf{A}_1) \cdot d\mathbf{a}_2 = \oint \mathbf{A}_1 \cdot d\mathbf{l}_2.$$

Now, according to Eq. 5.66,

$$\mathbf{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\mathbf{l}_1}{\mathscr{r}},$$

and hence

$$\Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint \left(\oint \frac{d\mathbf{l}_1}{\mathscr{r}} \right) \cdot d\mathbf{l}_2.$$

Evidently

$$M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{\mathscr{r}}. \tag{7.23}$$

This is the **Neumann formula**; it involves a double line integral—one integration around loop 1, the other around loop 2 (Fig. 7.31). It's not very useful for practical calculations, but it does reveal two important things about mutual inductance:

1. M_{21} is a purely geometrical quantity, having to do with the sizes, shapes, and relative positions of the two loops.

2. The integral in Eq. 7.23 is unchanged if we switch the roles of loops 1 and 2; it follows that

$$M_{21} = M_{12}. \tag{7.24}$$

This is an astonishing conclusion: *Whatever the shapes and positions of the loops, the flux through 2 when we run a current I around 1 is identical to the flux through 1 when we send the same current I around 2*. We may as well drop the subscripts and call them both M.

Example 7.10. A short solenoid (length l and radius a, with n_1 turns per unit length) lies on the axis of a very long solenoid (radius b, n_2 turns per unit length) as shown in Fig. 7.32. Current I flows in the short solenoid. What is the flux through the long solenoid?

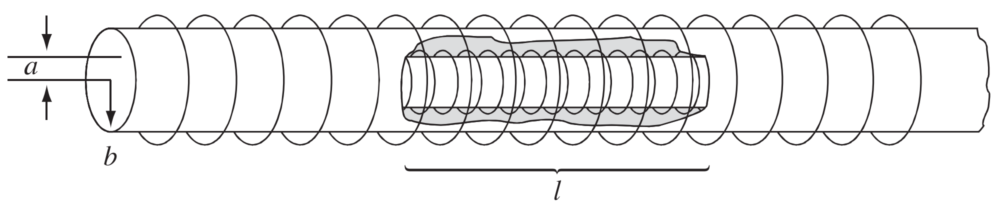

FIGURE 7.32

Solution

Since the inner solenoid is short, it has a very complicated field; moreover, it puts a different flux through each turn of the outer solenoid. It would be a *miserable* task to compute the total flux this way. However, if we exploit the equality of the mutual inductances, the problem becomes very easy. Just look at the reverse situation: run the current I through the *outer* solenoid, and calculate the flux through the *inner* one. The field inside the long solenoid is constant:

$$B = \mu_0 n_2 I$$

(Eq. 5.59), so the flux through a single loop of the short solenoid is

$$B\pi a^2 = \mu_0 n_2 I \pi a^2.$$

There are $n_1 l$ turns in all, so the total flux through the inner solenoid is

$$\Phi = \mu_0 \pi a^2 n_1 n_2 l I.$$

This is also the flux a current I in the *short* solenoid would put through the *long* one, which is what we set out to find. Incidentally, the mutual inductance, in this case, is

$$M = \mu_0 \pi a^2 n_1 n_2 l.$$

Suppose, now, that you *vary* the current in loop 1. The flux through loop 2 will vary accordingly, and Faraday's law says this changing flux will induce an emf in loop 2:

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M\frac{dI_1}{dt}. \tag{7.25}$$

(In quoting Eq. 7.22—which was based on the Biot-Savart law—I am tacitly assuming that the currents change slowly enough for the system to be considered quasistatic.) What a remarkable thing: Every time you change the current in loop 1, an induced current flows in loop 2—even though there are no wires connecting them!

Come to think of it, a changing current not only induces an emf in any nearby loops, it also induces an emf in the source loop *itself* (Fig 7.33). Once again, the field (and therefore also the flux) is proportional to the current:

$$\Phi = LI. \tag{7.26}$$

The constant of proportionality L is called the **self inductance** (or simply the **inductance**) of the loop. As with M, it depends on the geometry (size and shape) of the loop. If the current changes, the emf induced in the loop is

$$\mathcal{E} = -L\frac{dI}{dt}. \tag{7.27}$$

Inductance is measured in **henries** (H); a henry is a volt-second per ampere.

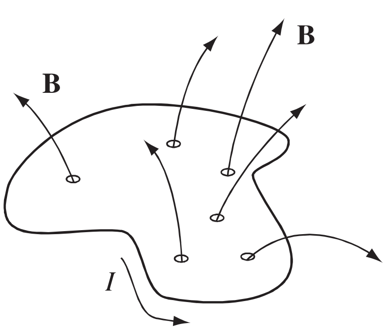

FIGURE 7.33

Example 7.11. Find the self-inductance of a toroidal coil with rectangular cross section (inner radius a, outer radius b, height h), that carries a total of N turns.

Solution
The magnetic field inside the toroid is (Eq. 5.60)

$$B = \frac{\mu_0 N I}{2\pi s}.$$

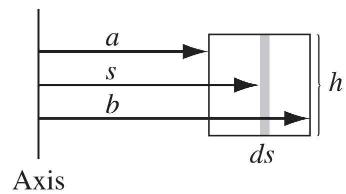

FIGURE 7.34

The flux through a single turn (Fig. 7.34) is

$$\int \mathbf{B} \cdot d\mathbf{a} = \frac{\mu_0 N I}{2\pi} h \int_a^b \frac{1}{s}\, ds = \frac{\mu_0 N I h}{2\pi} \ln\left(\frac{b}{a}\right).$$

The *total* flux is N times this, so the self-inductance (Eq. 7.26) is

$$L = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right). \tag{7.28}$$

Inductance (like capacitance) is an intrinsically *positive* quantity. Lenz's law, which is enforced by the minus sign in Eq. 7.27, dictates that the emf is in such a direction as to *oppose* any *change in current*. For this reason, it is called a **back emf**. Whenever you try to alter the current in a wire, you must fight against this back emf. Inductance plays somewhat the same role in electric circuits that *mass* plays in mechanical systems: The greater L is, the harder it is to change the current, just as the larger the mass, the harder it is to change an object's velocity.

Example 7.12. Suppose a current I is flowing around a loop, when someone suddenly cuts the wire. The current drops "instantaneously" to zero. This generates a whopping back emf, for although I may be small, dI/dt is enormous. (That's why you sometimes draw a spark when you unplug an iron or toaster—electromagnetic induction is desperately trying to keep the current going, even if it has to jump the gap in the circuit.)

Nothing so dramatic occurs when you plug *in* a toaster or iron. In this case induction opposes the sudden *increase* in current, prescribing instead a smooth and continuous buildup. Suppose, for instance, that a battery (which supplies a constant emf $\mathcal{E}_0$) is connected to a circuit of resistance R and inductance L (Fig. 7.35). What current flows?

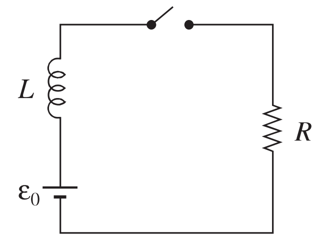

FIGURE 7.35

Solution

The total emf in this circuit is $\mathcal{E}_0$ from the battery plus $-L(dI/dt)$ from the inductance. Ohm's law, then, says[17]

$$\mathcal{E}_0 - L\frac{dI}{dt} = IR.$$

This is a first-order differential equation for I as a function of time. The general solution, as you can show for yourself, is

$$I(t) = \frac{\mathcal{E}_0}{R} + ke^{-(R/L)t},$$

where k is a constant to be determined by the initial conditions. In particular, if you close the switch at time $t = 0$, so $I(0) = 0$, then $k = -\mathcal{E}_0/R$, and

$$I(t) = \frac{\mathcal{E}_0}{R}\left[1 - e^{-(R/L)t}\right]. \tag{7.29}$$

This function is plotted in Fig. 7.36. Had there been no inductance in the circuit, the current would have jumped immediately to $\mathcal{E}_0/R$. In practice, *every* circuit has *some* self-inductance, and the current approaches $\mathcal{E}_0/R$ asymptotically. The quantity $\tau \equiv L/R$ is the **time constant**; it tells you how long the current takes to reach a substantial fraction (roughly two-thirds) of its final value.

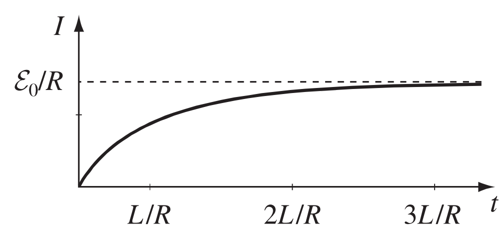

FIGURE 7.36

[17]Notice that $-L(dI/dt)$ goes on the *left* side of the equation—it is part of the emf that establishes the voltage across the resistor.

Problem 7.22 A small loop of wire (radius a) is held a distance z above the center of a large loop (radius b), as shown in Fig. 7.37. The planes of the two loops are parallel, and perpendicular to the common axis.

(a) Suppose current I flows in the big loop. Find the flux through the little loop. (The little loop is so small that you may consider the field of the big loop to be essentially constant.)

(b) Suppose current I flows in the little loop. Find the flux through the big loop. (The little loop is so small that you may treat it as a magnetic dipole.)

(c) Find the mutual inductances, and confirm that $M_{12} = M_{21}$.

Problem 7.23 A square loop of wire, of side a, lies midway between two long wires, $3a$ apart, and in the same plane. (Actually, the long wires are sides of a large rectangular loop, but the short ends are so far away that they can be neglected.) A clockwise current I in the square loop is gradually increasing: $dI/dt = k$ (a constant). Find the emf induced in the big loop. Which way will the induced current flow?

Problem 7.24 Find the self-inductance per unit length of a long solenoid, of radius R, carrying n turns per unit length.

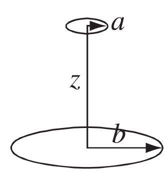

FIGURE 7.37

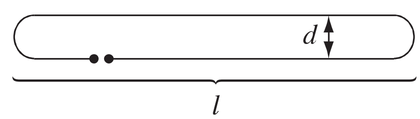

FIGURE 7.38

Problem 7.25 Try to compute the self-inductance of the "hairpin" loop shown in Fig. 7.38. (Neglect the contribution from the ends; most of the flux comes from the long straight section.) You'll run into a snag that is characteristic of many self-inductance calculations. To get a definite answer, assume the wire has a tiny radius ϵ, and ignore any flux through the wire itself.

Problem 7.26 An alternating current $I(t) = I_0 \cos(\omega t)$ (amplitude 0.5 A, frequency 60 Hz) flows down a straight wire, which runs along the axis of a toroidal coil with rectangular cross section (inner radius 1 cm, outer radius 2 cm, height 1 cm, 1000 turns). The coil is connected to a 500 Ω resistor.

(a) In the quasistatic approximation, what emf is induced in the toroid? Find the current, $I_R(t)$, in the resistor.

(b) Calculate the back emf in the coil, due to the current $I_R(t)$. What is the ratio of the amplitudes of this back emf and the "direct" emf in (a)?

Problem 7.27 A capacitor C is charged up to a voltage V and connected to an inductor L, as shown schematically in Fig. 7.39. At time $t = 0$, the switch S is closed. Find the current in the circuit as a function of time. How does your answer change if a resistor R is included in series with C and L?

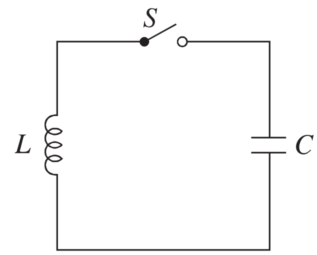

FIGURE 7.39

7.2.4 ■ Energy in Magnetic Fields

It takes a certain amount of energy to start a current flowing in a circuit. I'm not talking about the energy delivered to the resistors and converted into heat—that is irretrievably lost, as far as the circuit is concerned, and can be large or small, depending on how long you let the current run. What I am concerned with, rather, is the work you must do *against the back emf* to get the current going. This is a *fixed* amount, and it is *recoverable*: you get it back when the current is turned off. In the meantime, it represents energy latent in the circuit; as we'll see in a moment, it can be regarded as energy stored in the magnetic field.

The work done on a unit charge, against the back emf, in one trip around the circuit is $-\mathcal{E}$ (the minus sign records the fact that this is the work done *by you against* the emf, not the work done by the emf). The amount of charge per unit time passing down the wire is I. So the total work done per unit time is

$$\frac{dW}{dt} = -\mathcal{E}I = LI\frac{dI}{dt}.$$

If we start with zero current and build it up to a final value I, the work done (integrating the last equation over time) is

$$\boxed{W = \frac{1}{2}LI^2.} \tag{7.30}$$

It does not depend on how *long* we take to crank up the current, only on the geometry of the loop (in the form of L) and the final current I.

There is a nicer way to write W, which has the advantage that it is readily generalized to surface and volume currents. Remember that the flux Φ through the loop is equal to LI (Eq. 7.26). On the other hand,

$$\Phi = \int \mathbf{B} \cdot d\mathbf{a} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l},$$

where the line integral is around the perimeter of the loop. Thus

$$LI = \oint \mathbf{A} \cdot d\mathbf{l},$$

and therefore

$$W = \frac{1}{2} I \oint \mathbf{A} \cdot d\mathbf{l} = \frac{1}{2} \oint (\mathbf{A} \cdot \mathbf{I})\, dl. \tag{7.31}$$

In this form, the generalization to volume currents is obvious:

$$W = \frac{1}{2} \int_{\mathcal{V}} (\mathbf{A} \cdot \mathbf{J})\, d\tau. \tag{7.32}$$

But we can do even better, and express W entirely in terms of the magnetic field: Ampère's law, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, lets us eliminate $\mathbf{J}$:

$$W = \frac{1}{2\mu_0} \int \mathbf{A} \cdot (\nabla \times \mathbf{B})\, d\tau. \tag{7.33}$$

Integration by parts transfers the derivative from $\mathbf{B}$ to $\mathbf{A}$; specifically, product rule 6 states that

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}),$$

so

$$\mathbf{A} \cdot (\nabla \times \mathbf{B}) = \mathbf{B} \cdot \mathbf{B} - \nabla \cdot (\mathbf{A} \times \mathbf{B}).$$

Consequently,

$$\begin{aligned} W &= \frac{1}{2\mu_0} \left[\int B^2\, d\tau - \int \nabla \cdot (\mathbf{A} \times \mathbf{B})\, d\tau \right] \\ &= \frac{1}{2\mu_0} \left[\int_{\mathcal{V}} B^2\, d\tau - \oint_{\mathcal{S}} (\mathbf{A} \times \mathbf{B}) \cdot d\mathbf{a} \right], \end{aligned} \tag{7.34}$$

where $\mathcal{S}$ is the surface bounding the volume $\mathcal{V}$.

Now, the integration in Eq. 7.32 is to be taken over the *entire volume occupied by the current*. But any region *larger* than this will do just as well, for $\mathbf{J}$ is zero out there anyway. In Eq. 7.34, the larger the region we pick the greater is the contribution from the volume integral, and therefore the smaller is that of the surface integral (this makes sense: as the surface gets farther from the current, both $\mathbf{A}$ and $\mathbf{B}$ decrease). In particular, if we agree to integrate over *all* space, then the surface integral goes to zero, and we are left with

$$\boxed{W = \frac{1}{2\mu_0} \int_{\text{all space}} B^2\, d\tau.} \tag{7.35}$$

In view of this result, we say the energy is "stored in the magnetic field," in the amount $(B^2/2\mu_0)$ per unit volume. This is a nice way to think of it, though someone looking at Eq. 7.32 might prefer to say that the energy is stored in the *current distribution*, in the amount $\frac{1}{2}(\mathbf{A} \cdot \mathbf{J})$ per unit volume. The distinction is one of bookkeeping; the important quantity is the total energy W, and we need not worry about where (if anywhere) the energy is "located."

You might find it strange that it takes energy to set up a magnetic field—after all, magnetic fields *themselves* do no work. The point is that producing a magnetic field, where previously there was none, requires *changing* the field, and a changing **B**-field, according to Faraday, induces an *electric* field. The latter, of course, *can* do work. In the beginning, there is no **E**, and at the end there is no **E**; but in between, while **B** is building up, there *is* an **E**, and it is against *this* that the work is done. (You see why I could not calculate the energy stored in a magnetostatic field back in Chapter 5.) In the light of this, it is extraordinary how similar the magnetic energy formulas are to their electrostatic counterparts:[18]

$$W_{\text{elec}} = \frac{1}{2}\int (V\rho)\,d\tau = \frac{\epsilon_0}{2}\int E^2\,d\tau, \qquad \text{(2.43 and 2.45)}$$

$$W_{\text{mag}} = \frac{1}{2}\int (\mathbf{A}\cdot\mathbf{J})\,d\tau = \frac{1}{2\mu_0}\int B^2\,d\tau. \qquad \text{(7.32 and 7.35)}$$

Example 7.13. A long coaxial cable carries current I (the current flows down the surface of the inner cylinder, radius a, and back along the outer cylinder, radius b) as shown in Fig. 7.40. Find the magnetic energy stored in a section of length l.

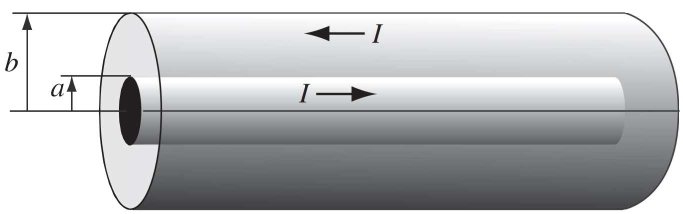

FIGURE 7.40

Solution

According to Ampère's law, the field between the cylinders is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}.$$

Elsewhere, the field is zero. Thus, the energy per unit volume is

$$\frac{1}{2\mu_0}\left(\frac{\mu_0 I}{2\pi s}\right)^2 = \frac{\mu_0 I^2}{8\pi^2 s^2}.$$

The energy in a cylindrical shell of length l, radius s, and thickness ds, then, is

$$\left(\frac{\mu_0 I^2}{8\pi^2 s^2}\right) 2\pi l s\,ds = \frac{\mu_0 I^2 l}{4\pi}\left(\frac{ds}{s}\right).$$

[18]For an illuminating confirmation of Eq. 7.35, using the method of Prob. 2.44, see T. H. Boyer, *Am. J. Phys.* **69**, 1 (2001).

Integrating from a to b, we have:

$$W = \frac{\mu_0 I^2 l}{4\pi} \ln\left(\frac{b}{a}\right).$$

By the way, this suggests a very simple way to calculate the self-inductance of the cable. According to Eq. 7.30, the energy can also be written as $\frac{1}{2}LI^2$. Comparing the two expressions,[19]

$$L = \frac{\mu_0 l}{2\pi} \ln\left(\frac{b}{a}\right).$$

This method of calculating self-inductance is especially useful when the current is not confined to a single path, but spreads over some surface or volume, so that different parts of the current enclose different amounts of flux. In such cases, it can be very tricky to get the inductance directly from Eq. 7.26, and it is best to let Eq. 7.30 *define* L.

Problem 7.28 Find the energy stored in a section of length l of a long solenoid (radius R, current I, n turns per unit length), (a) using Eq. 7.30 (you found L in Prob. 7.24); (b) using Eq. 7.31 (we worked out $\mathbf{A}$ in Ex. 5.12); (c) using Eq. 7.35; (d) using Eq. 7.34 (take as your volume the cylindrical tube from radius $a < R$ out to radius $b > R$).

Problem 7.29 Calculate the energy stored in the toroidal coil of Ex. 7.11, by applying Eq. 7.35. Use the answer to check Eq. 7.28.

Problem 7.30 A long cable carries current in one direction uniformly distributed over its (circular) cross section. The current returns along the surface (there is a very thin insulating sheath separating the currents). Find the self-inductance per unit length.

Problem 7.31 Suppose the circuit in Fig. 7.41 has been connected for a long time when suddenly, at time $t = 0$, switch S is thrown from A to B, bypassing the battery.

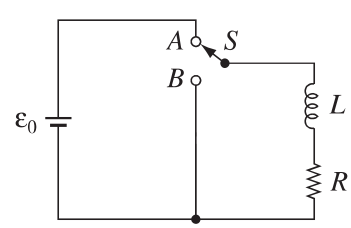

FIGURE 7.41

[19]Notice the similarity to Eq. 7.28—in a sense, the rectangular toroid *is* a short coaxial cable, turned on its side.

(a) What is the current at any subsequent time t?

(b) What is the total energy delivered to the resistor?

(c) Show that this is equal to the energy originally stored in the inductor.

Problem 7.32 Two tiny wire loops, with areas $\mathbf{a}_1$ and $\mathbf{a}_2$, are situated a displacement $\boldsymbol{\mathscr{r}}$ apart (Fig. 7.42).

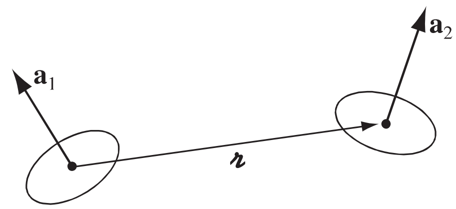

FIGURE 7.42

(a) Find their mutual inductance. [*Hint:* Treat them as magnetic dipoles, and use Eq. 5.88.] Is your formula consistent with Eq. 7.24?

(b) Suppose a current I_1 is flowing in loop 1, and we propose to turn on a current I_2 in loop 2. How much work must be done, against the mutually induced emf, to keep the current I_1 flowing in loop 1? In light of this result, comment on Eq. 6.35.

Problem 7.33 An infinite cylinder of radius R carries a uniform surface charge σ. We propose to set it spinning about its axis, at a final angular velocity ω_f. How much work will this take, per unit length? Do it two ways, and compare your answers:

(a) Find the magnetic field and the induced electric field (in the quasistatic approximation), inside and outside the cylinder, in terms of ω, $\dot{\omega}$, and s (the distance from the axis). Calculate the torque you must exert, and from that obtain the work done per unit length ($W = \int N\, d\phi$).

(b) Use Eq. 7.35 to determine the energy stored in the resulting magnetic field.

7.3 ■ MAXWELL'S EQUATIONS

7.3.1 ■ Electrodynamics Before Maxwell

So far, we have encountered the following laws, specifying the divergence and curl of electric and magnetic fields:

$$\begin{aligned}
&\text{(i)} && \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho && \text{(Gauss's law)},\\
&\text{(ii)} && \nabla \cdot \mathbf{B} = 0 && \text{(no name)},\\
&\text{(iii)} && \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} && \text{(Faraday's law)},\\
&\text{(iv)} && \nabla \times \mathbf{B} = \mu_0 \mathbf{J} && \text{(Ampère's law)}.
\end{aligned}$$

These equations represent the state of electromagnetic theory in the mid-nineteenth century, when Maxwell began his work. They were not written in so compact a form, in those days, but their physical content was familiar. Now, it happens that there is a fatal inconsistency in these formulas. It has to do with the old rule that divergence of curl is always zero. If you apply the divergence to number (iii), everything works out:

$$\nabla \cdot (\nabla \times \mathbf{E}) = \nabla \cdot \left(-\frac{\partial \mathbf{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}).$$

The left side is zero because divergence of curl is zero; the right side is zero by virtue of equation (ii). But when you do the same thing to number (iv), you get into trouble:

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0(\nabla \cdot \mathbf{J}); \tag{7.36}$$

the left side must be zero, but the right side, in general, is *not*. For *steady* currents, the divergence of $\mathbf{J}$ is zero, but when we go beyond magnetostatics Ampère's law cannot be right.

There's another way to see that Ampère's law is bound to fail for nonsteady currents. Suppose we're in the process of charging up a capacitor (Fig. 7.43). In integral form, Ampère's law reads

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}.$$

I want to apply it to the Amperian loop shown in the diagram. How do I determine I_{enc}? Well, it's the total current passing through the loop, or, more precisely, the current piercing a surface that has the loop for its boundary. In this case, the *simplest* surface lies in the plane of the loop—the wire punctures this surface, so $I_{\text{enc}} = I$. Fine—but what if I draw instead the balloon-shaped surface in Fig. 7.43? *No* current passes through *this* surface, and I conclude that $I_{\text{enc}} = 0$! We never had this problem in magnetostatics because the conflict arises only when charge

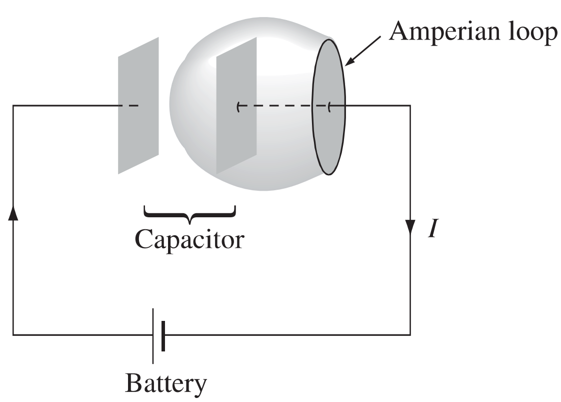

FIGURE 7.43

is piling up somewhere (in this case, on the capacitor plates). But for *nonsteady* currents (such as this one) "the current enclosed by the loop" is an ill-defined notion; it depends entirely on what surface you use. (If this seems pedantic to you—"obviously one should use the plane surface"—remember that the Amperian loop could be some contorted shape that doesn't even lie in a plane.)

Of course, we had no right to *expect* Ampère's law to hold outside of magnetostatics; after all, we derived it from the Biot-Savart law. However, in Maxwell's time there was no *experimental* reason to doubt that Ampère's law was of wider validity. The flaw was a purely theoretical one, and Maxwell fixed it by purely theoretical arguments.

7.3.2 ■ How Maxwell Fixed Ampère's Law

The problem is on the right side of Eq. 7.36, which *should be* zero, but *isn't*. Applying the continuity equation (5.29) and Gauss's law, the offending term can be rewritten:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot \mathbf{E}) = -\nabla \cdot \left(\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right).$$

If we were to combine $\epsilon_0(\partial \mathbf{E}/\partial t)$ with $\mathbf{J}$, in Ampère's law, it would be just right to kill off the extra divergence:

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.} \tag{7.37}$$

(Maxwell himself had other reasons for wanting to add this quantity to Ampère's law. To him, the rescue of the continuity equation was a happy dividend rather than a primary motive. But today we recognize this argument as a far more compelling one than Maxwell's, which was based on a now-discredited model of the ether.)[20]

Such a modification changes nothing, as far as magneto*statics* is concerned: when $\mathbf{E}$ is constant, we still have $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$. In fact, Maxwell's term is hard to detect in ordinary electromagnetic experiments, where it must compete for attention with $\mathbf{J}$—that's why Faraday and the others never discovered it in the laboratory. However, it plays a crucial role in the propagation of electromagnetic waves, as we'll see in Chapter 9.

Apart from curing the defect in Ampère's law, Maxwell's term has a certain aesthetic appeal: Just as a changing *magnetic* field induces an *electric* field (Faraday's law), so[21]

A changing electric field induces a magnetic field.

[20]For the history of this subject, see A. M. Bork, *Am. J. Phys.* **31**, 854 (1963).

[21]See footnote 8 (page 313) for commentary on the word "induce." The same issue arises here: Should a changing electric field be regarded as an independent source of magnetic field (along with current)? In a proximate sense it does function as a source, but since the electric field itself was produced by charges and currents, they alone are the "ultimate" sources of **E** and **B**. See S. E. Hill, *Phys. Teach.* **49**, 343 (2011); for a contrary view, see C. Savage, *Phys. Teach.* **50**, 226 (2012).

Of course, theoretical convenience and aesthetic consistency are only *suggestive*—there might, after all, be other ways to doctor up Ampère's law. The real confirmation of Maxwell's theory came in 1888 with Hertz's experiments on electromagnetic waves.

Maxwell called his extra term the **displacement current**:

$$\mathbf{J}_d \equiv \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{7.38}$$

(It's a misleading name; $\epsilon_0(\partial \mathbf{E}/\partial t)$ has nothing to do with current, except that it adds to $\mathbf{J}$ in Ampère's law.) Let's see now how displacement current resolves the paradox of the charging capacitor (Fig. 7.43). If the capacitor plates are very close together (I didn't *draw* them that way, but the calculation is simpler if you assume this), then the electric field between them is

$$E = \frac{1}{\epsilon_0}\sigma = \frac{1}{\epsilon_0}\frac{Q}{A},$$

where Q is the charge on the plate and A is its area. Thus, between the plates

$$\frac{\partial E}{\partial t} = \frac{1}{\epsilon_0 A}\frac{dQ}{dt} = \frac{1}{\epsilon_0 A} I.$$

Now, Eq. 7.37 reads, in integral form,

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \int \left(\frac{\partial \mathbf{E}}{\partial t}\right) \cdot d\mathbf{a}. \tag{7.39}$$

If we choose the *flat* surface, then $E = 0$ and $I_{\text{enc}} = I$. If, on the other hand, we use the balloon-shaped surface, then $I_{\text{enc}} = 0$, but $\int(\partial \mathbf{E}/\partial t) \cdot d\mathbf{a} = I/\epsilon_0$. So we get the same answer for either surface, though in the first case it comes from the conduction current, and in the second from the displacement current.

Example 7.14. Imagine two concentric metal spherical shells (Fig. 7.44).

The inner one (radius a) carries a charge $Q(t)$, and the outer one (radius b) an opposite charge $-Q(t)$. The space between them is filled with Ohmic material of conductivity σ, so a radial current flows:

$$\mathbf{J} = \sigma \mathbf{E} = \sigma \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}; \quad I = -\dot{Q} = \int \mathbf{J} \cdot d\mathbf{a} = \frac{\sigma Q}{\epsilon_0}.$$

This configuration is spherically symmetrical, so the magnetic field has to be zero (the only direction it could possibly point is radial, and $\nabla \cdot \mathbf{B} = 0 \Rightarrow \oint \mathbf{B} \cdot d\mathbf{a} = B(4\pi r^2) = 0$, so $\mathbf{B} = \mathbf{0}$). *What?* I thought currents produce magnetic fields! Isn't that what Biot-Savart and Ampère taught us? How can there be a $\mathbf{J}$ with no accompanying $\mathbf{B}$?

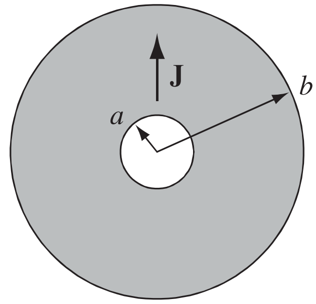

FIGURE 7.44

Solution

This is not a static configuration: Q, $\mathbf{E}$, and $\mathbf{J}$ are all functions of time; Ampère and Biot-Savart do not apply. The displacement current

$$J_d = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{4\pi} \frac{\dot{Q}}{r^2} \hat{\mathbf{r}} = -\sigma \frac{Q}{4\pi \epsilon_0 r^2} \hat{\mathbf{r}}$$

exactly cancels the conduction current (in Eq. 7.37), and the magnetic field (determined by $\nabla \cdot \mathbf{B} = 0$, $\nabla \times \mathbf{B} = \mathbf{0}$) is indeed zero.

Problem 7.34 A fat wire, radius a, carries a constant current I, uniformly distributed over its cross section. A narrow gap in the wire, of width $w \ll a$, forms a parallel-plate capacitor, as shown in Fig. 7.45. Find the magnetic field in the gap, at a distance $s < a$ from the axis.

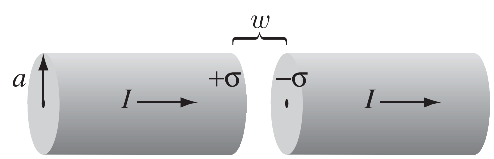

FIGURE 7.45

Problem 7.35 The preceding problem was an artificial model for the charging capacitor, designed to avoid complications associated with the current spreading out over the surface of the plates. For a more realistic model, imagine *thin* wires that connect to the centers of the plates (Fig. 7.46a). Again, the current I is constant, the radius of the capacitor is a, and the separation of the plates is $w \ll a$. Assume that the current flows out over the plates in such a way that the surface charge is uniform, at any given time, and is zero at $t = 0$.

(a) Find the electric field between the plates, as a function of t.

(b) Find the displacement current through a circle of radius s in the plane midway between the plates. Using this circle as your "Amperian loop," and the flat surface that spans it, find the magnetic field at a distance s from the axis.

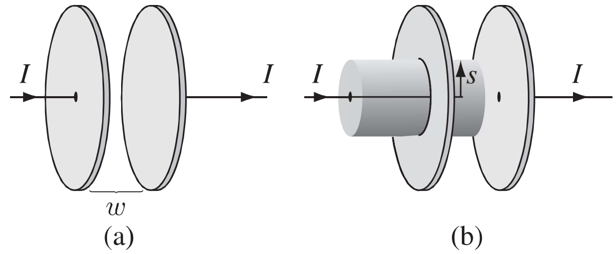

FIGURE 7.46

(c) Repeat part (b), but this time use the cylindrical surface in Fig. 7.46(b), which is open at the right end and extends to the left through the plate and terminates outside the capacitor. Notice that the displacement current through this surface is zero, and there are two contributions to I_{enc}.[22]

Problem 7.36 Refer to Prob. 7.16, to which the correct answer was

$$\mathbf{E}(s, t) = \frac{\mu_0 I_0 \omega}{2\pi} \sin(\omega t) \ln\left(\frac{a}{s}\right) \hat{\mathbf{z}}.$$

(a) Find the displacement current density $\mathbf{J}_d$.

(b) Integrate it to get the total displacement current,

$$I_d = \int \mathbf{J}_d \cdot d\mathbf{a}.$$

(c) Compare I_d and I. (What's their ratio?) If the outer cylinder were, say, 2 mm in diameter, how high would the frequency have to be, for I_d to be 1% of I? [This problem is designed to indicate why Faraday never discovered displacement currents, and why it is ordinarily safe to ignore them unless the frequency is extremely high.]

7.3.3 ■ Maxwell's Equations

In the last section we put the finishing touches on Maxwell's equations:

$$\begin{aligned}
&\text{(i)} \quad \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho && \text{(Gauss's law)},\\
&\text{(ii)} \quad \nabla \cdot \mathbf{B} = 0 && \text{(no name)},\\
&\text{(iii)} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} && \text{(Faraday's law)},\\
&\text{(iv)} \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} && \text{(Ampère's law with Maxwell's correction)}.
\end{aligned} \tag{7.40}$$

[22]This problem raises an interesting quasi-philosophical question: If you measure **B** in the laboratory, have you detected the effects of displacement current (as (b) would suggest), or merely confirmed the effects of ordinary currents (as (c) implies)? See D. F. Bartlett, *Am. J. Phys.* **58**, 1168 (1990).

Together with the force law,

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{7.41}$$

they summarize the entire theoretical content of classical electrodynamics[23] (save for some special properties of matter, which we encountered in Chapters 4 and 6). Even the continuity equation,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}, \tag{7.42}$$

which is the mathematical expression of conservation of charge, can be derived from Maxwell's equations by applying the divergence to number (iv).

I have written Maxwell's equations in the traditional way, which emphasizes that they specify the divergence and curl of **E** and **B**. In this form, they reinforce the notion that electric fields can be produced *either* by charges (ρ) *or* by changing magnetic fields ($\partial \mathbf{B}/\partial t$), and magnetic fields can be produced *either* by currents (**J**) *or* by changing electric fields ($\partial \mathbf{E}/\partial t$). Actually, this is misleading, because $\partial \mathbf{B}/\partial t$ and $\partial \mathbf{E}/\partial t$ are *themselves* due to charges and currents. I think it is logically preferable to write

$$\left.\begin{array}{ll} \text{(i)}\ \ \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho, & \text{(iii)}\ \ \nabla \times \mathbf{E} + \dfrac{\partial \mathbf{B}}{\partial t} = \mathbf{0}, \\[2ex] \text{(ii)}\ \ \nabla \cdot \mathbf{B} = 0, & \text{(iv)}\ \ \nabla \times \mathbf{B} - \mu_0\epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}, \end{array}\right\} \tag{7.43}$$

with the fields (**E** and **B**) on the left and the sources (ρ and **J**) on the right. This notation emphasizes that all electromagnetic fields are ultimately attributable to charges and currents. Maxwell's equations tell you how *charges* produce *fields*; reciprocally, the force law tells you how *fields* affect *charges*.

Problem 7.37 Suppose

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\theta(vt - r)\hat{\mathbf{r}}; \quad \mathbf{B}(\mathbf{r}, t) = \mathbf{0}$$

(The theta function is defined in Prob. 1.46b). Show that these fields satisfy all of Maxwell's equations, and determine ρ and **J**. Describe the physical situation that gives rise to these fields.

7.3.4 ■ Magnetic Charge

There is a pleasing symmetry to Maxwell's equations; it is particularly striking in free space, where ρ and **J** vanish:

$$\left.\begin{array}{ll} \nabla \cdot \mathbf{E} = 0, & \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}, \\[2ex] \nabla \cdot \mathbf{B} = 0, & \nabla \times \mathbf{B} = \mu_0\epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}. \end{array}\right\}$$

[23] Like any differential equations, Maxwell's must be supplemented by suitable *boundary conditions*. Because these are typically "obvious" from the context (e.g. **E** and **B** go to zero at large distances from a localized charge distribution), it is easy to forget that they play an essential role.

If you replace $\mathbf{E}$ by $\mathbf{B}$ and $\mathbf{B}$ by $-\mu_0\epsilon_0\mathbf{E}$, the first pair of equations turns into the second, and vice versa. This symmetry[24] between $\mathbf{E}$ and $\mathbf{B}$ is spoiled, though, by the charge term in Gauss's law and the current term in Ampère's law. You can't help wondering why the corresponding quantities are "missing" from $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$. What if we had

$$\left.\begin{array}{ll} \text{(i)}\ \ \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho_e, & \text{(iii)}\ \ \nabla \times \mathbf{E} = -\mu_0\mathbf{J}_m - \dfrac{\partial \mathbf{B}}{\partial t}, \\ \text{(ii)}\ \ \nabla \cdot \mathbf{B} = \mu_0\rho_m, & \text{(iv)}\ \ \nabla \times \mathbf{B} = \mu_0\mathbf{J}_e + \mu_0\epsilon_0\dfrac{\partial \mathbf{E}}{\partial t}. \end{array}\right\} \quad (7.44)$$

Then ρ_m would represent the density of magnetic "charge," and ρ_e the density of electric charge; $\mathbf{J}_m$ would be the current of magnetic charge, and $\mathbf{J}_e$ the current of electric charge. Both charges would be conserved:

$$\nabla \cdot \mathbf{J}_m = -\frac{\partial \rho_m}{\partial t}, \quad \text{and} \quad \nabla \cdot \mathbf{J}_e = -\frac{\partial \rho_e}{\partial t}. \quad (7.45)$$

The former follows by application of the divergence to (iii), the latter by taking the divergence of (iv).

In a sense, Maxwell's equations *beg* for magnetic charge to exist—it would fit in so nicely. And yet, in spite of a diligent search, no one has ever found any.[25] As far as we know, ρ_m is zero everywhere, and so is $\mathbf{J}_m$; $\mathbf{B}$ is *not* on equal footing with $\mathbf{E}$: there exist stationary sources for $\mathbf{E}$ (electric charges) but none for $\mathbf{B}$. (This is reflected in the fact that magnetic multipole expansions have no monopole term, and magnetic dipoles consist of current loops, not separated north and south "poles.") Apparently God just didn't *make* any magnetic charge. (In *quantum* electrodynamics, by the way, it's a more than merely aesthetic shame that magnetic charge does not seem to exist: Dirac showed that the existence of *magnetic* charge would explain why *electric* charge is *quantized*. See Prob. 8.19.)

Problem 7.38 Assuming that "Coulomb's law" for magnetic charges (q_m) reads

$$\mathbf{F} = \frac{\mu_0}{4\pi}\frac{q_{m_1}q_{m_2}}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}, \quad (7.46)$$

work out the force law for a monopole q_m moving with velocity $\mathbf{v}$ through electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$.[26]

Problem 7.39 Suppose a magnetic monopole q_m passes through a resistanceless loop of wire with self-inductance L. What current is induced in the loop?[27]

[24] Don't be distracted by the pesky constants μ_0 and ϵ_0; these are present only because the SI system measures $\mathbf{E}$ and $\mathbf{B}$ in different units, and would not occur, for instance, in the Gaussian system.

[25] For an extensive bibliography, see A. S. Goldhaber and W. P. Trower, *Am. J. Phys.* **58**, 429 (1990).

[26] For interesting commentary, see W. Rindler, *Am. J. Phys.* **57**, 993 (1989).

[27] This is one of the methods used to search for monopoles in the laboratory; see B. Cabrera, *Phys. Rev. Lett.* **48**, 1378 (1982).

7.3.5 ■ Maxwell's Equations in Matter

Maxwell's equations in the form 7.40 are complete and correct as they stand. However, when you are working with materials that are subject to electric and magnetic polarization there is a more convenient way to *write* them. For inside polarized matter there will be accumulations of "bound" charge and current, over which you exert no direct control. It would be nice to reformulate Maxwell's equations so as to make explicit reference only to the "free" charges and currents.

We have already learned, from the static case, that an electric polarization **P** produces a bound charge density

$$\rho_b = -\nabla \cdot \mathbf{P} \tag{7.47}$$

(Eq. 4.12). Likewise, a magnetic polarization (or "magnetization") **M** results in a bound current

$$\mathbf{J}_b = \nabla \times \mathbf{M} \tag{7.48}$$

(Eq. 6.13). There's just one new feature to consider in the *non*static case: Any *change* in the electric polarization involves a flow of (bound) charge (call it $\mathbf{J}_p$), which must be included in the total current. For suppose we examine a tiny chunk of polarized material (Fig. 7.47). The polarization introduces a charge density $\sigma_b = P$ at one end and $-\sigma_b$ at the other (Eq. 4.11). If P now *increases* a bit, the charge on each end increases accordingly, giving a net current

$$dI = \frac{\partial \sigma_b}{\partial t} da_{\perp} = \frac{\partial P}{\partial t} da_{\perp}.$$

The current density, therefore, is

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}. \tag{7.49}$$

This **polarization current** has nothing to do with the *bound* current $\mathbf{J}_b$. The latter is associated with *magnetization* of the material and involves the spin and orbital motion of electrons; $\mathbf{J}_p$, by contrast, is the result of the linear motion of charge when the electric polarization changes. If **P** points to the right, and is increasing, then each plus charge moves a bit to the right and each minus charge to the left; the cumulative effect is the polarization current $\mathbf{J}_p$. We ought to check that Eq. 7.49 is consistent with the continuity equation:

$$\nabla \cdot \mathbf{J}_p = \nabla \cdot \frac{\partial \mathbf{P}}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{P}) = -\frac{\partial \rho_b}{\partial t}.$$

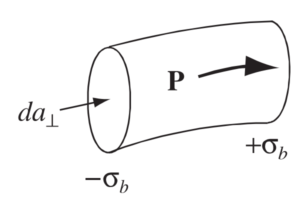

FIGURE 7.47

Yes: The continuity equation *is* satisfied; in fact, $\mathbf{J}_p$ is essential to ensure the conservation of bound charge. (Incidentally, a changing *magnetization* does *not* lead to any analogous accumulation of charge or current. The bound current $\mathbf{J}_b = \nabla \times \mathbf{M}$ varies in response to changes in $\mathbf{M}$, to be sure, but that's about it.)

In view of all this, the total charge density can be separated into two parts:

$$\rho = \rho_f + \rho_b = \rho_f - \nabla \cdot \mathbf{P}, \tag{7.50}$$

and the current density into *three* parts:

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b + \mathbf{J}_p = \mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}. \tag{7.51}$$

Gauss's law can now be written as

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}(\rho_f - \nabla \cdot \mathbf{P}),$$

or

$$\nabla \cdot \mathbf{D} = \rho_f, \tag{7.52}$$

where, as in the static case,

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}. \tag{7.53}$$

Meanwhile, Ampère's law (with Maxwell's term) becomes

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \right) + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t},$$

or

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}, \tag{7.54}$$

where, as before,

$$\mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}. \tag{7.55}$$

Faraday's law and $\nabla \cdot \mathbf{B} = 0$ are not affected by our separation of charge and current into free and bound parts, since they do not involve ρ or $\mathbf{J}$.

In terms of *free* charges and currents, then, Maxwell's equations read

$$\boxed{\begin{array}{ll} \text{(i)}\ \ \nabla \cdot \mathbf{D} = \rho_f, & \text{(iii)}\ \ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}, \\[2ex] \text{(ii)}\ \ \nabla \cdot \mathbf{B} = 0, & \text{(iv)}\ \ \nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t}. \end{array}} \tag{7.56}$$

Some people regard these as the "true" Maxwell's equations, but please understand that they are in *no way* more "general" than Eq. 7.40; they simply reflect a convenient division of charge and current into free and nonfree parts. And they have the disadvantage of hybrid notation, since they contain both **E** and **D**, both **B** and **H**. They must be supplemented, therefore, by appropriate **constitutive relations**, giving **D** and **H** in terms of **E** and **B**. These depend on the nature of the material; for linear media

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \text{and} \quad \mathbf{M} = \chi_m \mathbf{H}, \tag{7.57}$$

so

$$\mathbf{D} = \epsilon \mathbf{E}, \quad \text{and} \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B}, \tag{7.58}$$

where $\epsilon \equiv \epsilon_0(1 + \chi_e)$ and $\mu \equiv \mu_0(1 + \chi_m)$. Incidentally, you'll remember that **D** is called the electric "displacement"; that's why the second term in the Ampère/Maxwell equation (iv) came to be called the **displacement current**. In this context,

$$\mathbf{J}_d \equiv \frac{\partial \mathbf{D}}{\partial t}. \tag{7.59}$$

Problem 7.40 Sea water at frequency $\nu = 4 \times 10^8$ Hz has permittivity $\epsilon = 81\epsilon_0$, permeability $\mu = \mu_0$, and resistivity $\rho = 0.23\ \Omega \cdot \text{m}$. What is the ratio of conduction current to displacement current? [*Hint:* Consider a parallel-plate capacitor immersed in sea water and driven by a voltage $V_0 \cos(2\pi\nu t)$.]

7.3.6 ■ Boundary Conditions

In general, the fields **E, B, D,** and **H** will be discontinuous at a boundary between two different media, or at a surface that carries a charge density σ or a current density **K**. The explicit form of these discontinuities can be deduced from Maxwell's equations (7.56), in their integral form

$$\left.\begin{array}{ll} \text{(i)} & \oint_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}} \\ \text{(ii)} & \oint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a} = 0 \end{array}\right\} \text{ over any closed surface } \mathcal{S}.$$

$$\left.\begin{array}{ll} \text{(iii)} & \oint_{\mathcal{P}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a} \\ \text{(iv)} & \oint_{\mathcal{P}} \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}} + \frac{d}{dt}\int_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{a} \end{array}\right\} \begin{array}{l}\text{for any surface } \mathcal{S} \\ \text{bounded by the} \\ \text{closed loop } \mathcal{P}.\end{array}$$

Applying (i) to a tiny, wafer-thin Gaussian pillbox extending just slightly into the material on either side of the boundary (Fig. 7.48), we obtain:

$$\mathbf{D}_1 \cdot \mathbf{a} - \mathbf{D}_2 \cdot \mathbf{a} = \sigma_f\, a.$$

(The positive direction for **a** is *from* 2 *toward* 1. The edge of the wafer contributes nothing in the limit as the thickness goes to zero; nor does any *volume*

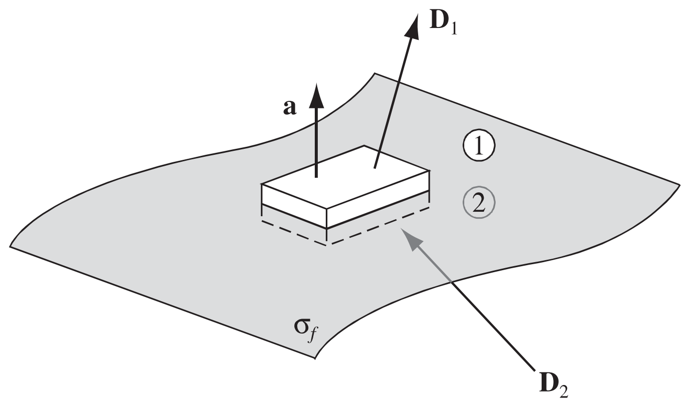

FIGURE 7.48

charge density.) Thus, the component of **D** that is perpendicular to the interface is discontinuous in the amount

$$\boxed{D_1^{\perp} - D_2^{\perp} = \sigma_f.} \tag{7.60}$$

Identical reasoning, applied to equation (ii), yields

$$\boxed{B_1^{\perp} - B_2^{\perp} = 0.} \tag{7.61}$$

Turning to (iii), a very thin Amperian loop straddling the surface gives

$$\mathbf{E}_1 \cdot \mathbf{l} - \mathbf{E}_2 \cdot \mathbf{l} = -\frac{d}{dt}\int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a}.$$

But in the limit as the width of the loop goes to zero, the flux vanishes. (I have already dropped the contribution of the two ends to $\oint \mathbf{E} \cdot d\mathbf{l}$, on the same grounds.) Therefore,

$$\boxed{\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0}.} \tag{7.62}$$

That is, the components of **E** *parallel* to the interface are continuous across the boundary. By the same token, (iv) implies

$$\mathbf{H}_1 \cdot \mathbf{l} - \mathbf{H}_2 \cdot \mathbf{l} = I_{f_{\text{enc}}},$$

where $I_{f_{\text{enc}}}$ is the free current passing through the Amperian loop. No *volume* current density will contribute (in the limit of infinitesimal width), but a *surface* current can. In fact, if $\hat{\mathbf{n}}$ is a unit vector perpendicular to the interface (pointing from 2 toward 1), so that $(\hat{\mathbf{n}} \times \mathbf{l})$ is normal to the Amperian loop (Fig. 7.49), then

$$I_{f_{\text{enc}}} = \mathbf{K}_f \cdot (\hat{\mathbf{n}} \times \mathbf{l}) = (\mathbf{K}_f \times \hat{\mathbf{n}}) \cdot \mathbf{l},$$

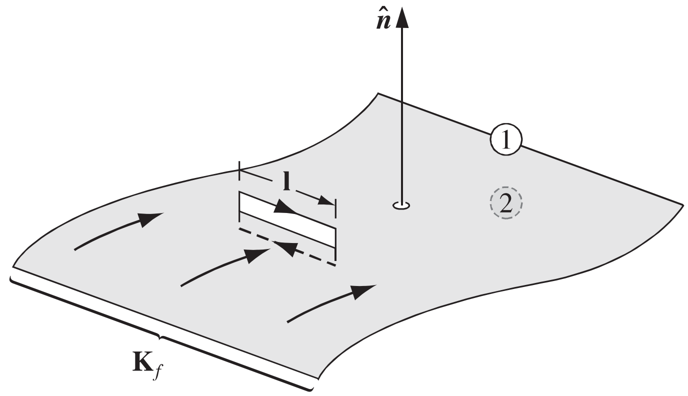

FIGURE 7.49

and hence

$$\boxed{\mathbf{H}_1^{\parallel} - \mathbf{H}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}.} \tag{7.63}$$

So the *parallel* components of **H** are discontinuous by an amount proportional to the free surface current density.

Equations 7.60-63 are the general boundary conditions for electrodynamics. In the case of *linear* media, they can be expressed in terms of **E** and **B** alone:

$$\left.\begin{array}{ll} \text{(i)}\ \ \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f, & \text{(iii)}\ \ \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0}, \\ \text{(ii)}\ \ B_1^{\perp} - B_2^{\perp} = 0, & \text{(iv)}\ \ \dfrac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \dfrac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}. \end{array}\right\} \tag{7.64}$$

In particular, if there is no free charge or free current at the interface, then

$$\begin{array}{ll} \text{(i)}\ \ \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = 0, & \text{(iii)}\ \ \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0}, \\ \text{(ii)}\ \ B_1^{\perp} - B_2^{\perp} = 0, & \text{(iv)}\ \ \dfrac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \dfrac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{0}. \end{array} \tag{7.65}$$

As we shall see in Chapter 9, these equations are the basis for the theory of reflection and refraction.

More Problems on Chapter 7

! **Problem 7.41** Two long, straight copper pipes, each of radius a, are held a distance $2d$ apart (see Fig. 7.50). One is at potential V_0, the other at $-V_0$. The space surrounding the pipes is filled with weakly conducting material of conductivity σ. Find the current per unit length that flows from one pipe to the other. [*Hint:* Refer to Prob. 3.12.]

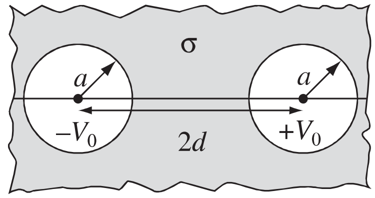

FIGURE 7.50

! **Problem 7.42** A rare case in which the electrostatic field **E** for a circuit can actually be *calculated* is the following:[28] Imagine an infinitely long cylindrical sheet, of uniform resistivity and radius a. A slot (corresponding to the battery) is maintained at $\pm V_0/2$, at $\phi = \pm\pi$, and a steady current flows over the surface, as indicated in Fig. 7.51. According to Ohm's law, then,

$$V(a, \phi) = \frac{V_0 \phi}{2\pi}, \quad (-\pi < \phi < +\pi).$$

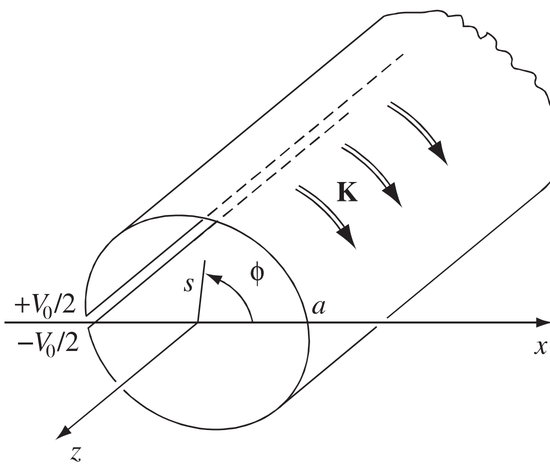

FIGURE 7.51

(a) Use separation of variables in cylindrical coordinates to determine $V(s, \phi)$ inside and outside the cylinder. [*Answer:* $(V_0/\pi)\tan^{-1}[(s\sin\phi)/(a + s\cos\phi)]$, $(s < a)$; $(V_0/\pi)\tan^{-1}[(a\sin\phi)/(s + a\cos\phi)]$, $(s > a)$]

(b) Find the surface charge density on the cylinder. [*Answer:* $(\epsilon_0 V_0/\pi a)\tan(\phi/2)$]

Problem 7.43 The magnetic field outside a long straight wire carrying a steady current I is

$$\mathbf{B} = \frac{\mu_0}{2\pi}\frac{I}{s}\,\hat{\boldsymbol{\phi}}.$$

The *electric field inside* the wire is uniform:

$$\mathbf{E} = \frac{I\rho}{\pi a^2}\,\hat{\mathbf{z}},$$

[28] M. A. Heald, *Am. J. Phys.* **52**, 522 (1984). See also J. A. Hernandes and A. K. T. Assis, *Phys. Rev. E* **68**, 046611 (2003).

where ρ is the resistivity and a is the radius (see Exs. 7.1 and 7.3). *Question:* What is the electric field *outside* the wire?[29] The answer depends on how you complete the circuit. Suppose the current returns along a perfectly conducting grounded coaxial cylinder of radius b (Fig. 7.52). In the region $a < s < b$, the potential $V(s, z)$ satisfies Laplace's equation, with the boundary conditions

$$\text{(i)}\ \ V(a, z) = -\frac{I\rho z}{\pi a^2}; \quad \text{(ii)}\ \ V(b, z) = 0.$$

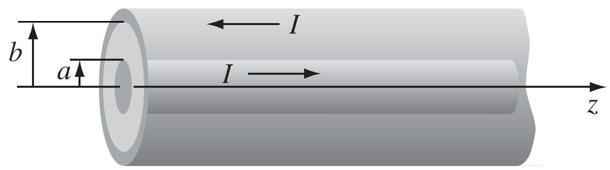

FIGURE 7.52

This does not suffice to determine the answer—we still need to specify boundary conditions at the two ends (though for a *long* wire it shouldn't matter much). In the literature, it is customary to sweep this ambiguity under the rug by simply *stipulating* that $V(s, z)$ is proportional to z: $V(s, z) = zf(s)$. On this assumption:

(a) Determine $f(s)$.

(b) Find $\mathbf{E}(s, z)$.

(c) Calculate the surface charge density $\sigma(z)$ on the wire.

[*Answer:* $V = (-Iz\rho/\pi a^2)[\ln(s/b)/\ln(a/b)]$ This is a peculiar result, since E_s and $\sigma(z)$ are *not* independent of z—as one would certainly expect for a truly *infinite* wire.]

Problem 7.44 In a **perfect conductor**, the conductivity is infinite, so $\mathbf{E} = \mathbf{0}$ (Eq. 7.3), and any net charge resides on the surface (just as it does for an *im*perfect conductor, in electro*statics*).

(a) Show that the magnetic field is constant ($\partial\mathbf{B}/\partial t = \mathbf{0}$), inside a perfect conductor.

(b) Show that the magnetic flux through a perfectly conducting loop is constant.

A **superconductor** is a perfect conductor with the additional property that the (constant) **B** inside is in fact *zero*. (This "flux exclusion" is known as the **Meissner effect**.[30])

[29]This is a famous problem, first analyzed by Sommerfeld, and is known in its most recent incarnation as **Merzbacher's puzzle**. A. Sommerfeld, *Electrodynamics*, p. 125 (New York: Academic Press, 1952); E. Merzbacher, *Am. J. Phys.* **48**, 178 (1980); further references in R. N. Varnay and L. H. Fisher, *Am. J. Phys.* **52**, 1097 (1984).

[30]The Meissner effect is sometimes referred to as "perfect diamagnetism," in the sense that the field inside is not merely *reduced*, but canceled entirely. However, the surface currents responsible for this are *free*, not bound, so the actual *mechanism* is quite different.

(c) Show that the current in a superconductor is confined to the surface.

(d) Superconductivity is lost above a certain critical temperature (T_c), which varies from one material to another. Suppose you had a sphere (radius a) above its critical temperature, and you held it in a uniform magnetic field $B_0\hat{\mathbf{z}}$ while cooling it below T_c. Find the induced surface current density $\mathbf{K}$, as a function of the polar angle θ.

Problem 7.45 A familiar demonstration of superconductivity (Prob. 7.44) is the levitation of a magnet over a piece of superconducting material. This phenomenon can be analyzed using the method of images.[31] Treat the magnet as a perfect dipole $\mathbf{m}$, a height z above the origin (and constrained to point in the z direction), and pretend that the superconductor occupies the entire half-space below the xy plane. Because of the Meissner effect, $\mathbf{B} = \mathbf{0}$ for $z \leq 0$, and since $\mathbf{B}$ is divergenceless, the normal (z) component is continuous, so $B_z = 0$ just *above* the surface. This boundary condition is met by the image configuration in which an identical dipole is placed at $-z$, as a stand-in for the superconductor; the two arrangements therefore produce the same magnetic field in the region $z > 0$.

(a) Which way should the image dipole point ($+z$ or $-z$)?

(b) Find the force on the magnet due to the induced currents in the superconductor (which is to say, the force due to the image dipole). Set it equal to Mg (where M is the mass of the magnet) to determine the height h at which the magnet will "float." [*Hint:* Refer to Prob. 6.3.]

(c) The induced current on the surface of the superconductor (the xy plane) can be determined from the boundary condition on the *tangential* component of $\mathbf{B}$ (Eq. 5.76): $\mathbf{B} = \mu_0(\mathbf{K} \times \hat{\mathbf{z}})$. Using the field you get from the image configuration, show that

$$\mathbf{K} = -\frac{3mrh}{2\pi(r^2 + h^2)^{5/2}}\,\hat{\boldsymbol{\phi}},$$

where r is the distance from the origin.

! **Problem 7.46** If a magnetic dipole levitating above an infinite superconducting plane (Prob. 7.45) is free to rotate, what orientation will it adopt, and how high above the surface will it float?

Problem 7.47 A perfectly conducting spherical shell of radius a rotates about the z axis with angular velocity ω, in a uniform magnetic field $\mathbf{B} = B_0\,\hat{\mathbf{z}}$. Calculate the emf developed between the "north pole" and the equator. [*Answer:* $\frac{1}{2}B_0\omega a^2$]

! **Problem 7.48** Refer to Prob. 7.11 (and use the result of Prob. 5.42): How long does is take a falling *circular* ring (radius a, mass m, resistance R) to cross the bottom of the magnetic field B, at its (changing) terminal velocity?

[31] W. M. Saslow, *Am. J. Phys.* **59**, 16 (1991).

Problem 7.49

(a) Referring to Prob. 5.52(a) and Eq. 7.18, show that

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}, \tag{7.66}$$

for Faraday-induced electric fields. Check this result by taking the divergence and curl of both sides.

(b) A spherical shell of radius R carries a uniform surface charge σ. It spins about a fixed axis at an angular velocity $\omega(t)$ that changes slowly with time. Find the electric field inside and outside the sphere. [*Hint:* There are *two* contributions here: the Coulomb field due to the charge, and the Faraday field due to the changing **B**. Refer to Ex. 5.11.]

Problem 7.50 Electrons undergoing cyclotron motion can be sped up by increasing the magnetic field; the accompanying electric field will impart tangential acceleration. This is the principle of the **betatron**. One would like to keep the radius of the orbit constant during the process. Show that this can be achieved by designing a magnet such that the average field over the area of the orbit is twice the field at the circumference (Fig. 7.53). Assume the electrons start from rest in zero field, and that the apparatus is symmetric about the center of the orbit. (Assume also that the electron velocity remains well below the speed of light, so that nonrelativistic mechanics applies.) [*Hint:* Differentiate Eq. 5.3 with respect to time, and use $F = ma = qE$.]

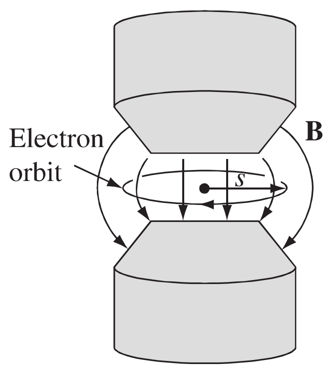

FIGURE 7.53

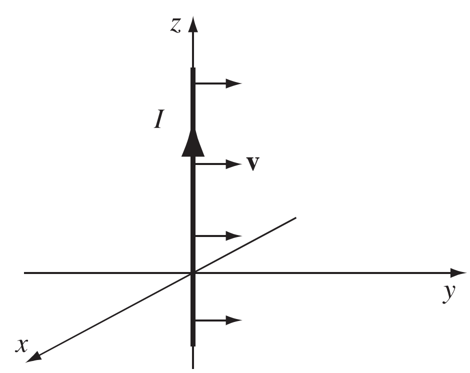

FIGURE 7.54

Problem 7.51 An infinite wire carrying a constant current I in the $\hat{\mathbf{z}}$ direction is moving in the y direction at a constant speed v. Find the electric field, in the quasistatic approximation, at the instant the wire coincides with the z axis (Fig. 7.54). [*Answer:* $-(\mu_0 I v/2\pi s)\sin\phi\,\hat{\mathbf{z}}$]

Problem 7.52 An atomic electron (charge q) circles about the nucleus (charge Q) in an orbit of radius r; the centripetal acceleration is provided, of course, by the Coulomb attraction of opposite charges. Now a small magnetic field dB is slowly turned on, perpendicular to the plane of the orbit. Show that the increase in kinetic energy, dT, imparted by the induced electric field, is just right to sustain circular motion *at the same radius* r. (That's why, in my discussion of diamagnetism, I assumed the radius is fixed. See Sect. 6.1.3 and the references cited there.)

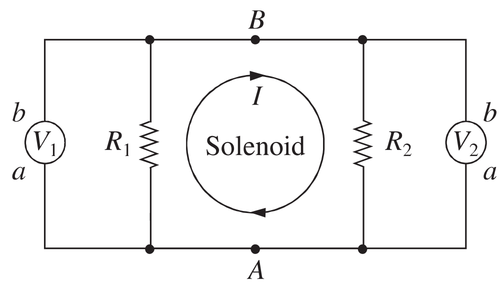

FIGURE 7.55

Problem 7.53 The current in a long solenoid is increasing linearly with time, so the flux is proportional to t: $\Phi = \alpha t$. Two voltmeters are connected to diametrically opposite points (A and B), together with resistors (R_1 and R_2), as shown in Fig. 7.55. What is the reading on each voltmeter? Assume that these are *ideal* voltmeters that draw negligible current (they have huge internal resistance), and that a voltmeter registers $-\int_a^b \mathbf{E} \cdot d\mathbf{l}$ between the terminals and through the meter. [*Answer:* $V_1 = \alpha R_1/(R_1 + R_2)$; $V_2 = -\alpha R_2/(R_1 + R_2)$. Notice that $V_1 \neq V_2$, even though they are connected to the same points![32]]

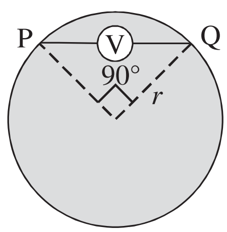

FIGURE 7.56

Problem 7.54 A circular wire loop (radius r, resistance R) encloses a region of uniform magnetic field, B, perpendicular to its plane. The field (occupying the shaded region in Fig. 7.56) increases linearly with time ($B = \alpha t$). An ideal voltmeter (infinite internal resistance) is connected between points P and Q.

(a) What is the current in the loop?

(b) What does the voltmeter read? [*Answer:* $\alpha r^2/2$]

Problem 7.55 In the discussion of motional emf (Sect. 7.1.3) I assumed that the wire loop (Fig. 7.10) has a resistance R; the current generated is then $I = vBh/R$. But what if the wire is made out of perfectly conducting material, so that R is *zero*? In that case, the current is limited only by the back emf associated with the self-inductance L of the loop (which would ordinarily be negligible in comparison with IR). Show that in this régime the loop (mass m) executes simple harmonic motion, and find its frequency.[33] [*Answer:* $\omega = Bh/\sqrt{mL}$]

[32]R. H. Romer, *Am. J. Phys.* **50**, 1089 (1982). See also H. W. Nicholson, *Am. J. Phys.* **73**, 1194 (2005); B. M. McGuyer, *Am. J. Phys.* **80**, 101 (2012).

[33]For a collection of related problems, see W. M. Saslow, *Am. J. Phys.* **55**, 986 (1987), and R. H. Romer, *Eur. J. Phys.* **11**, 103 (1990).

Problem 7.56

(a) Use the Neumann formula (Eq. 7.23) to calculate the mutual inductance of the configuration in Fig. 7.37, assuming a is very small ($a \ll b$, $a \ll z$). Compare your answer to Prob. 7.22.

(b) For the general case (*not* assuming a is small), show that

$$M = \frac{\mu_0 \pi \beta}{2} \sqrt{ab\beta} \left(1 + \frac{15}{8}\beta^2 + \ldots\right),$$

where

$$\beta \equiv \frac{ab}{z^2 + a^2 + b^2}.$$

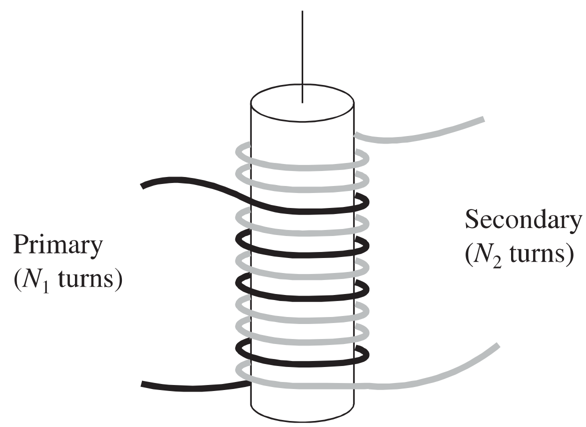

FIGURE 7.57

Problem 7.57 Two coils are wrapped around a cylindrical form in such a way that the *same flux passes through every turn of both coils*. (In practice this is achieved by inserting an iron core through the cylinder; this has the effect of concentrating the flux.) The **primary** coil has N_1 turns and the **secondary** has N_2 (Fig. 7.57). If the current I in the primary is changing, show that the emf in the secondary is given by

$$\frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{N_2}{N_1}, \tag{7.67}$$

where $\mathcal{E}_1$ is the (back) emf of the primary. [This is a primitive **transformer**—a device for raising or lowering the emf of an alternating current source. By choosing the appropriate number of turns, any desired secondary emf can be obtained. If you think this violates the conservation of energy, study Prob. 7.58.]

Problem 7.58 A transformer (Prob. 7.57) takes an input AC voltage of amplitude V_1, and delivers an output voltage of amplitude V_2, which is determined by the turns ratio ($V_2/V_1 = N_2/N_1$). If $N_2 > N_1$, the output voltage is greater than the input voltage. Why doesn't this violate conservation of energy? *Answer:* Power is the product of voltage and current; if the voltage goes *up*, the current must come *down*. The purpose of this problem is to see exactly how this works out, in a simplified model.

(a) In an ideal transformer, the same flux passes through all turns of the primary and of the secondary. Show that in this case $M^2 = L_1 L_2$, where M is the mutual inductance of the coils, and L_1, L_2 are their individual self-inductances.

(b) Suppose the primary is driven with AC voltage $V_{\text{in}} = V_1 \cos(\omega t)$, and the secondary is connected to a resistor, R. Show that the two currents satisfy the relations

$$L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt} = V_1 \cos(\omega t); \quad L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt} = -I_2 R.$$

(c) Using the result in (a), solve these equations for $I_1(t)$ and $I_2(t)$. (Assume I_1 has no DC component.)

(d) Show that the output voltage ($V_{\text{out}} = I_2 R$) divided by the input voltage (V_{in}) is equal to the turns ratio: $V_{\text{out}}/V_{\text{in}} = N_2/N_1$.

(e) Calculate the input power ($P_{\text{in}} = V_{\text{in}} I_1$) and the output power ($P_{\text{out}} = V_{\text{out}} I_2$), and show that their averages over a full cycle are equal.

Problem 7.59 An infinite wire runs along the z axis; it carries a current $I(z)$ that is a function of z (but not of t), and a charge density $\lambda(t)$ that is a function of t (but not of z).

(a) By examining the charge flowing into a segment dz in a time dt, show that $d\lambda/dt = -dI/dz$. If we stipulate that $\lambda(0) = 0$ and $I(0) = 0$, show that $\lambda(t) = kt$, $I(z) = -kz$, where k is a constant.

(b) Assume for a moment that the process is quasistatic, so the fields are given by Eqs. 2.9 and 5.38. Show that these are in fact the *exact* fields, by confirming that all four of Maxwell's equations are satisfied. (First do it in differential form, for the region $s > 0$, then in integral form for the appropriate Gaussian cylinder/Amperian loop straddling the axis.)

Problem 7.60 Suppose $\mathbf{J}(\mathbf{r})$ is constant in time but $\rho(\mathbf{r}, t)$ is *not*—conditions that might prevail, for instance, during the charging of a capacitor.

(a) Show that the charge density at any particular point is a linear function of time:

$$\rho(\mathbf{r}, t) = \rho(\mathbf{r}, 0) + \dot{\rho}(\mathbf{r}, 0)t,$$

where $\dot{\rho}(\mathbf{r}, 0)$ is the time derivative of ρ at $t = 0$. [*Hint:* Use the continuity equation.]

This is *not* an electrostatic or magnetostatic configuration;[34] nevertheless, rather surprisingly, both Coulomb's law (Eq. 2.8) and the Biot-Savart law (Eq. 5.42) hold, as you can confirm by showing that they satisfy Maxwell's equations. In particular:

[34]Some authors *would* regard this as magnetostatic, since $\mathbf{B}$ is independent of t. For them, the Biot-Savart law is a general rule of magnetostatics, but $\nabla \cdot \mathbf{J} = 0$ and $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ apply only under the *additional* assumption that ρ is constant. In such a formulation, Maxwell's displacement term can (in this very special case) be *derived* from the Biot-Savart law, by the method of part (b). See D. F. Bartlett, *Am. J. Phys.* **58**, 1168 (1990); D. J. Griffiths and M. A. Heald, *Am. J. Phys.* **59**, 111 (1991).

(b) Show that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \, d\tau'$$

obeys Ampère's law *with Maxwell's displacement current term.*

Problem 7.61 The magnetic field of an infinite straight wire carrying a steady current I can be obtained from the *displacement* current term in the Ampère/Maxwell law, as follows: Picture the current as consisting of a uniform line charge λ moving along the z axis at speed v (so that $I = \lambda v$), with a tiny gap of length ϵ, which reaches the origin at time $t = 0$. In the next instant (up to $t = \epsilon/v$) there is no *real* current passing through a circular Amperian loop in the xy plane, but there *is* a *displacement* current, due to the "missing" charge in the gap.

(a) Use Coulomb's law to calculate the z component of the electric field, for points in the xy plane a distance s from the origin, due to a segment of wire with uniform density $-\lambda$ extending from $z_1 = vt - \epsilon$ to $z_2 = vt$.

(b) Determine the flux of this electric field through a circle of radius a in the xy plane.

(c) Find the displacement current through this circle. Show that I_d is equal to I, in the limit as the gap width (ϵ) goes to zero.[35]

Problem 7.62 A certain transmission line is constructed from two thin metal "ribbons," of width w, a very small distance $h \ll w$ apart. The current travels down one strip and back along the other. In each case, it spreads out uniformly over the surface of the ribbon.

(a) Find the capacitance per unit length, $\mathcal{C}$.

(b) Find the inductance per unit length, $\mathcal{L}$.

(c) What is the product $\mathcal{LC}$, numerically? [$\mathcal{L}$ and $\mathcal{C}$ will, of course, vary from one kind of transmission line to another, but their *product* is a universal constant—check, for example, the cable in Ex. 7.13—provided the space between the conductors is a vacuum. In the theory of transmission lines, this product is related to the speed with which a pulse propagates down the line: $v = 1/\sqrt{\mathcal{LC}}$.]

(d) If the strips are insulated from one another by a nonconducting material of permittivity ϵ and permeability μ, what then is the product $\mathcal{LC}$? What is the propagation speed? [*Hint:* see Ex. 4.6; by what factor does L change when an inductor is immersed in linear material of permeability μ?]

Problem 7.63 Prove **Alfven's theorem**: In a perfectly conducting fluid (say, a gas of free electrons), the magnetic flux through any closed loop moving with the fluid is constant in time. (The magnetic field lines are, as it were, "frozen" into the fluid.)

(a) Use Ohm's law, in the form of Eq. 7.2, together with Faraday's law, to prove that if $\sigma = \infty$ and $\mathbf{J}$ is finite, then

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}).$$

[35] For a slightly different approach to the same problem, see W. K. Terry, *Am. J. Phys.* **50**, 742 (1982).

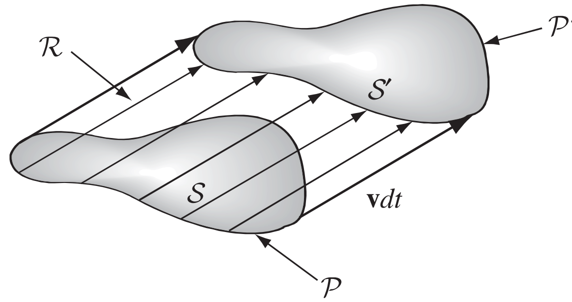

FIGURE 7.58

(b) Let $\mathcal{S}$ be the surface bounded by the loop ($\mathcal{P}$) at time t, and $\mathcal{S}'$ a surface bounded by the loop in its new position ($\mathcal{P}'$) at time $t + dt$ (see Fig. 7.58). The change in flux is

$$d\Phi = \int_{\mathcal{S}'} \mathbf{B}(t + dt) \cdot d\mathbf{a} - \int_{\mathcal{S}} \mathbf{B}(t) \cdot d\mathbf{a}.$$

Use $\nabla \cdot \mathbf{B} = 0$ to show that

$$\int_{\mathcal{S}'} \mathbf{B}(t + dt) \cdot d\mathbf{a} + \int_{\mathcal{R}} \mathbf{B}(t + dt) \cdot d\mathbf{a} = \int_{\mathcal{S}} \mathbf{B}(t + dt) \cdot d\mathbf{a}$$

(where $\mathcal{R}$ is the "ribbon" joining $\mathcal{P}$ and $\mathcal{P}'$), and hence that

$$d\Phi = dt \int_{\mathcal{S}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} - \int_{\mathcal{R}} \mathbf{B}(t + dt) \cdot d\mathbf{a}$$

(for infinitesimal dt). Use the method of Sect. 7.1.3 to rewrite the second integral as

$$dt \oint_{\mathcal{P}} (\mathbf{B} \times \mathbf{v}) \cdot d\mathbf{l},$$

and invoke Stokes' theorem to conclude that

$$\frac{d\Phi}{dt} = \int_{\mathcal{S}} \left(\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) \right) \cdot d\mathbf{a}.$$

Together with the result in (a), this proves the theorem.

Problem 7.64

(a) Show that Maxwell's equations with magnetic charge (Eq. 7.44) are invariant under the **duality transformation**

$$\left. \begin{aligned} \mathbf{E}' &= \mathbf{E} \cos\alpha + c\mathbf{B} \sin\alpha, \\ c\mathbf{B}' &= c\mathbf{B} \cos\alpha - \mathbf{E} \sin\alpha, \\ cq_e' &= cq_e \cos\alpha + q_m \sin\alpha, \\ q_m' &= q_m \cos\alpha - cq_e \sin\alpha, \end{aligned} \right\} \tag{7.68}$$

where $c \equiv 1/\sqrt{\epsilon_0\mu_0}$ and α is an arbitrary rotation angle in "**E/B**-space." Charge and current densities transform in the same way as q_e and q_m. [This means, in particular, that if you know the fields produced by a configuration of *electric* charge, you can immediately (using $\alpha = 90^\circ$) write down the fields produced by the corresponding arrangement of *magnetic* charge.]

(b) Show that the force law (Prob. 7.38)

$$\mathbf{F} = q_e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + q_m\left(\mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E}\right) \tag{7.69}$$

is also invariant under the duality transformation.

Intermission

All of our cards are now on the table, and in a sense my job is done. In the first seven chapters we assembled electrodynamics piece by piece, and now, with Maxwell's equations in their final form, the theory is complete. There are no more laws to be learned, no further generalizations to be considered, and (with perhaps one exception) no lurking inconsistencies to be resolved. If yours is a one-semester course, this would be a reasonable place to stop.

But in another sense we have just arrived at the starting point. We are at last in possession of a full deck—it's time to deal. This is the fun part, in which one comes to appreciate the extraordinary power and richness of electrodynamics. In a full-year course there should be plenty of time to cover the remaining chapters, and perhaps to supplement them with a unit on plasma physics, say, or AC circuit theory, or even a little general relativity. But if you have room for only one topic, I'd recommend Chapter 9, on Electromagnetic Waves (you'll probably want to skim Chapter 8 as preparation). This is the segue to Optics, and is historically the most important application of Maxwell's theory.

CHAPTER

8 Conservation Laws

8.1 ■ CHARGE AND ENERGY

8.1.1 ■ The Continuity Equation

In this chapter we study conservation of energy, momentum, and angular momentum, in electrodynamics. But I want to begin by reviewing the conservation of *charge*, because it is the paradigm for all conservation laws. What precisely does conservation of charge tell us? That the total charge in the universe is constant? Well, sure—that's **global** conservation of charge. But **local** conservation of charge is a much stronger statement: If the charge in some region changes, then exactly that amount of charge must have passed in or out through the surface. The tiger can't simply rematerialize outside the cage; if it got from inside to outside it must have slipped through a hole in the fence.

Formally, the charge in a volume $\mathcal{V}$ is

$$Q(t) = \int_{\mathcal{V}} \rho(\mathbf{r}, t)\, d\tau, \tag{8.1}$$

and the current flowing out through the boundary $\mathcal{S}$ is $\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}$, so local conservation of charge says

$$\frac{dQ}{dt} = -\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}. \tag{8.2}$$

Using Eq. 8.1 to rewrite the left side, and invoking the divergence theorem on the right, we have

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t}\, d\tau = -\int_{\mathcal{V}} \nabla \cdot \mathbf{J}\, d\tau, \tag{8.3}$$

and since this is true for *any* volume, it follows that

$$\boxed{\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}.} \tag{8.4}$$

This is the continuity equation—the precise mathematical statement of local conservation of charge. It can be derived from Maxwell's equations—conservation of charge is not an *independent* assumption; it is built into the laws of electrodynamics. It serves as a constraint on the sources (ρ and $\mathbf{J}$). They can't be just *any* old functions—they have to respect conservation of charge.[1]

The purpose of this chapter is to develop the corresponding equations for local conservation of energy and momentum. In the process (and perhaps more important) we will learn how to express the energy density and the momentum density (the analogs to ρ), as well as the energy "current" and the momentum "current" (analogous to $\mathbf{J}$).

8.1.2 ■ Poynting's Theorem

In Chapter 2, we found that the work necessary to assemble a static charge distribution (against the Coulomb repulsion of like charges) is (Eq. 2.45)

$$W_e = \frac{\epsilon_0}{2}\int E^2\,d\tau,$$

where $\mathbf{E}$ is the resulting electric field. Likewise, the work required to get currents going (against the back emf) is (Eq. 7.35)

$$W_m = \frac{1}{2\mu_0}\int B^2\,d\tau,$$

where $\mathbf{B}$ is the resulting magnetic field. This suggests that the total energy stored in electromagnetic fields, per unit volume, is

$$\boxed{u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right).} \tag{8.5}$$

In this section I will confirm Eq. 8.5, and develop the energy conservation law for electrodynamics.

Suppose we have some charge and current configuration which, at time t, produces fields $\mathbf{E}$ and $\mathbf{B}$. In the next instant, dt, the charges move around a bit. *Question*: How much work, dW, is done by the electromagnetic forces acting on these charges, in the interval dt? According to the Lorentz force law, the work done on a charge q is

$$\mathbf{F}\cdot d\mathbf{l} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\mathbf{v}\,dt = q\mathbf{E}\cdot\mathbf{v}\,dt.$$

In terms of the charge and current densities, $q \to \rho d\tau$ and $\rho\mathbf{v} \to \mathbf{J}$,[2] so the rate at which work is done on all the charges in a volume $\mathcal{V}$ is

$$\frac{dW}{dt} = \int_{\mathcal{V}} (\mathbf{E}\cdot\mathbf{J})\,d\tau. \tag{8.6}$$

[1]The continuity equation is the *only* such constraint. Any functions $\rho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$ consistent with Eq. 8.4 constitute possible charge and current densities, in the sense of admitting solutions to Maxwell's equations.

[2]This is a slippery equation: after all, if charges of both signs are present, the *net* charge density can be zero even when the current is *not*—in fact, this is the case for ordinary current-carrying wires. We should really treat the positive and negative charges separately, and combine the two to get Eq. 8.6, with $\mathbf{J} = \rho_+\mathbf{v}_+ + \rho_-\mathbf{v}_-$.

Evidently $\mathbf{E}\cdot\mathbf{J}$ is the work done per unit time, per unit volume—which is to say, the *power* delivered per unit volume. We can express this quantity in terms of the fields alone, using the Ampère-Maxwell law to eliminate $\mathbf{J}$:

$$\mathbf{E}\cdot\mathbf{J} = \frac{1}{\mu_0}\mathbf{E}\cdot(\nabla\times\mathbf{B}) - \epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}.$$

From product rule 6,

$$\nabla\cdot(\mathbf{E}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{B}).$$

Invoking Faraday's law ($\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$), it follows that

$$\mathbf{E}\cdot(\nabla\times\mathbf{B}) = -\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} - \nabla\cdot(\mathbf{E}\times\mathbf{B}).$$

Meanwhile,

$$\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(B^2), \quad \text{and} \quad \mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(E^2), \tag{8.7}$$

so

$$\mathbf{E}\cdot\mathbf{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\nabla\cdot(\mathbf{E}\times\mathbf{B}). \tag{8.8}$$

Putting this into Eq. 8.6, and applying the divergence theorem to the second term, we have

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}}\frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau - \frac{1}{\mu_0}\oint_{\mathcal{S}}(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}, \tag{8.9}$$

where $\mathcal{S}$ is the surface bounding $\mathcal{V}$. This is **Poynting's theorem**; it is the "work-energy theorem" of electrodynamics. The first integral on the right is the total energy stored in the fields, $\int u\,d\tau$ (Eq. 8.5). The second term evidently represents the rate at which energy is transported out of $\mathcal{V}$, across its boundary surface, by the electromagnetic fields. Poynting's theorem says, then, that *the work done on the charges by the electromagnetic force is equal to the decrease in energy remaining in the fields, less the energy that flowed out through the surface*.

The *energy per unit time, per unit area*, transported by the fields is called the **Poynting vector**:

$$\boxed{\mathbf{S} \equiv \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}).} \tag{8.10}$$

Specifically, $\mathbf{S}\cdot d\mathbf{a}$ is the energy per unit time crossing the infinitesimal surface $d\mathbf{a}$—the energy *flux* (so $\mathbf{S}$ is the **energy flux density**).[3] We will see many

[3]If you're very fastidious, you'll notice a small gap in the logic here: We know from Eq. 8.9 that $\oint\mathbf{S}\cdot d\mathbf{a}$ is the total power passing through a *closed* surface, but this does not prove that $\int\mathbf{S}\cdot d\mathbf{a}$ is the power passing through any *open* surface (there could be an extra term that integrates to zero over all closed surfaces). This is, however, the obvious and natural interpretation; as always, the precise *location* of energy is not really determined in electrodynamics (see Sect. 2.4.4).

applications of the Poynting vector in Chapters 9 and 11, but for the moment I am mainly interested in using it to express Poynting's theorem more compactly:

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} u\, d\tau - \oint_{\mathcal{S}} \mathbf{S}\cdot d\mathbf{a}. \tag{8.11}$$

What if *no* work is done on the charges in $\mathcal{V}$—what if, for example, we are in a region of empty space, where there *is* no charge? In that case $dW/dt = 0$, so

$$\int \frac{\partial u}{\partial t}\, d\tau = -\oint \mathbf{S}\cdot d\mathbf{a} = -\int (\nabla\cdot\mathbf{S})\, d\tau,$$

and hence

$$\frac{\partial u}{\partial t} = -\nabla\cdot\mathbf{S}. \tag{8.12}$$

This is the "continuity equation" for *energy*—u (energy density) plays the role of ρ (charge density), and $\mathbf{S}$ takes the part of $\mathbf{J}$ (current density). It expresses local conservation of electromagnetic energy.

In *general*, though, electromagnetic energy by itself is *not* conserved (nor is the energy of the charges). Of course not! The fields do work on the charges, and the charges create fields—energy is tossed back and forth between them. In the overall energy economy, you must include the contributions of both the matter and the fields.

Example 8.1. When current flows down a wire, work is done, which shows up as Joule heating of the wire (Eq. 7.7). Though there are certainly *easier* ways to do it, the energy per unit time delivered to the wire can be calculated using the Poynting vector. Assuming it's uniform, the electric field parallel to the wire is

$$E = \frac{V}{L},$$

where V is the potential difference between the ends and L is the length of the wire (Fig. 8.1). The magnetic field is "circumferential"; at the surface (radius a) it has the value

$$B = \frac{\mu_0 I}{2\pi a}.$$

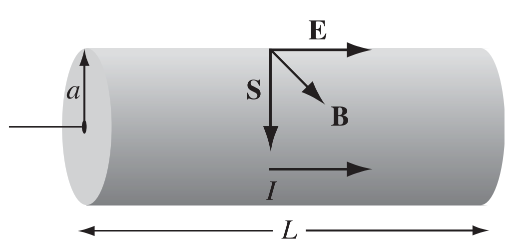

FIGURE 8.1

Accordingly, the magnitude of the Poynting vector is

$$S = \frac{1}{\mu_0}\frac{V}{L}\frac{\mu_0 I}{2\pi a} = \frac{VI}{2\pi a L},$$

and it points radially inward. The energy per unit time passing in through the surface of the wire is therefore

$$\int \mathbf{S}\cdot d\mathbf{a} = S(2\pi a L) = VI,$$

which is exactly what we concluded, on much more direct grounds, in Sect. 7.1.1.[4]

Problem 8.1 Calculate the power (energy per unit time) transported down the cables of Ex. 7.13 and Prob. 7.62, assuming the two conductors are held at potential difference V, and carry current I (down one and back up the other).

Problem 8.2 Consider the charging capacitor in Prob. 7.34.

(a) Find the electric and magnetic fields in the gap, as functions of the distance s from the axis and the time t. (Assume the charge is zero at $t = 0$.)

(b) Find the energy density u_{em} and the Poynting vector $\mathbf{S}$ in the gap. Note especially the *direction* of $\mathbf{S}$. Check that Eq. 8.12 is satisfied.

(c) Determine the total energy in the gap, as a function of time. Calculate the total power flowing into the gap, by integrating the Poynting vector over the appropriate surface. Check that the power input is equal to the rate of increase of energy in the gap (Eq. 8.9—in this case $W = 0$, because there is no charge in the gap). [If you're worried about the fringing fields, do it for a volume of radius $b < a$ well inside the gap.]

8.2 ■ MOMENTUM

8.2.1 ■ Newton's Third Law in Electrodynamics

Imagine a point charge q traveling in along the x axis at a constant speed v. Because it is moving, its electric field is *not* given by Coulomb's law; nevertheless, $\mathbf{E}$ still points radially outward from the instantaneous position of the charge (Fig. 8.2a), as we'll see in Chapter 10. Since, moreover, a moving point charge does not constitute a steady current, its magnetic field is *not* given by the Biot-Savart law. Nevertheless, it's a fact that $\mathbf{B}$ still circles around the axis in a manner suggested by the right-hand rule (Fig. 8.2b); again, the proof will come in Chapter 10.

[4] What about energy flow *down* the wire? For a discussion, see M. K. Harbola, *Am. J. Phys.* **78**, 1203 (2010). For a more sophisticated geometry, see B. S. Davis and L. Kaplan, *Am. J. Phys.* **79**, 1155 (2011).

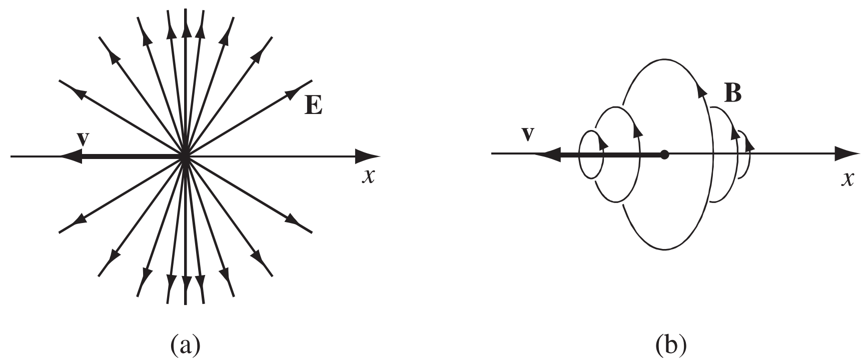

FIGURE 8.2

Now suppose this charge encounters an identical one, proceeding in at the same speed along the y axis. Of course, the electromagnetic force between them would tend to drive them off the axes, but let's assume that they're mounted on tracks, or something, so they're obliged to maintain the same direction and the same speed (Fig. 8.3). The electric force between them is repulsive, but how about the magnetic force? Well, the magnetic field of q_1 points into the page (at the position of q_2), so the magnetic force on q_2 is toward the *right*, whereas the magnetic field of q_2 is *out* of the page (at the position of q_1), and the magnetic force on q_1 is *upward*. *The net electromagnetic force of q_1 on q_2 is equal but not opposite to the force of q_2 on q_1, in violation of Newton's third law*. In electro*statics* and magneto*statics* the third law holds, but in electro*dynamics* it does *not*.

Well, that's an interesting curiosity, but then, how often does one actually use the third law, in practice? *Answer:* All the time! For the proof of conservation of momentum rests on the cancellation of internal forces, which follows from the third law. When you tamper with the third law, you are placing conservation of momentum in jeopardy, and there is hardly any principle in physics more sacred than *that*.

Momentum conservation is rescued, in electrodynamics, by the realization that the *fields themselves carry momentum*. This is not so surprising when you

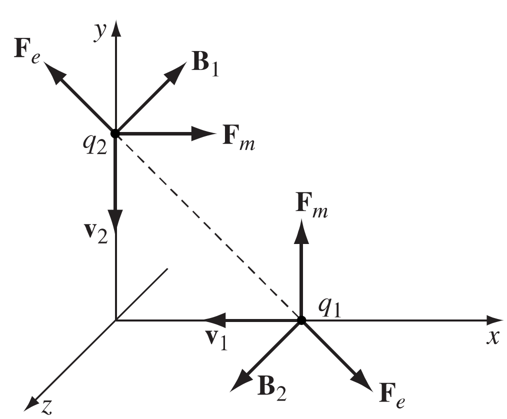

FIGURE 8.3

consider that we have already attributed *energy* to the fields. Whatever momentum is lost to the particles is gained by the fields. Only when the field momentum is added to the mechanical momentum is momentum conservation restored.

8.2.2 ■ Maxwell's Stress Tensor

Let's calculate the total electromagnetic force on the charges in volume $\mathcal{V}$:

$$\mathbf{F} = \int_{\mathcal{V}} (\mathbf{E} + \mathbf{v} \times \mathbf{B})\rho \, d\tau = \int_{\mathcal{V}} (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}) \, d\tau. \tag{8.13}$$

The *force per unit volume* is

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}. \tag{8.14}$$

As before, I propose to express this in terms of fields alone, eliminating ρ and $\mathbf{J}$ by using Maxwell's equations (i) and (iv):

$$\mathbf{f} = \epsilon_0(\nabla \cdot \mathbf{E})\mathbf{E} + \left(\frac{1}{\mu_0}\nabla \times \mathbf{B} - \epsilon_0\frac{\partial \mathbf{E}}{\partial t}\right) \times \mathbf{B}.$$

Now

$$\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) = \left(\frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}\right) + \left(\mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t}\right),$$

and Faraday's law says

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E},$$

so

$$\frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} = \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) + \mathbf{E} \times (\nabla \times \mathbf{E}).$$

Thus

$$\mathbf{f} = \epsilon_0 \left[(\nabla \cdot \mathbf{E})\mathbf{E} - \mathbf{E} \times (\nabla \times \mathbf{E})\right] - \frac{1}{\mu_0}\left[\mathbf{B} \times (\nabla \times \mathbf{B})\right] - \epsilon_0\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}). \tag{8.15}$$

Just to make things look more symmetrical, let's throw in a term $(\nabla \cdot \mathbf{B})\mathbf{B}$; since $\nabla \cdot \mathbf{B} = 0$, this costs us nothing. Meanwhile, product rule 4 says

$$\nabla(E^2) = 2(\mathbf{E} \cdot \nabla)\mathbf{E} + 2\mathbf{E} \times (\nabla \times \mathbf{E}),$$

so

$$\mathbf{E} \times (\nabla \times \mathbf{E}) = \frac{1}{2}\nabla(E^2) - (\mathbf{E} \cdot \nabla)\mathbf{E},$$

and the same goes for **B**. Therefore,

$$\begin{aligned}\mathbf{f} = \epsilon_0 \left[(\nabla \cdot \mathbf{E})\mathbf{E} + (\mathbf{E} \cdot \nabla)\mathbf{E}\right] + \frac{1}{\mu_0}\left[(\nabla \cdot \mathbf{B})\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{B}\right] \\ -\frac{1}{2}\nabla\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \epsilon_0 \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}).\end{aligned} \tag{8.16}$$

Ugly! But it can be simplified by introducing the **Maxwell stress tensor**,

$$T_{ij} \equiv \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2\right). \tag{8.17}$$

The indices i and j refer to the coordinates x, y, and z, so the stress tensor has a total of nine components (T_{xx}, T_{yy}, T_{xz}, T_{yx}, and so on). The **Kronecker delta**, δ_{ij}, is 1 if the indices are the same ($\delta_{xx} = \delta_{yy} = \delta_{zz} = 1$) and zero otherwise ($\delta_{xy} = \delta_{xz} = \delta_{yz} = 0$). Thus

$$T_{xx} = \frac{1}{2}\epsilon_0\left(E_x^2 - E_y^2 - E_z^2\right) + \frac{1}{2\mu_0}\left(B_x^2 - B_y^2 - B_z^2\right),$$

$$T_{xy} = \epsilon_0(E_x E_y) + \frac{1}{\mu_0}(B_x B_y),$$

and so on.

Because it carries *two* indices, where a vector has only one, T_{ij} is sometimes written with a double arrow: $\overleftrightarrow{\mathbf{T}}$. One can form the dot product of $\overleftrightarrow{\mathbf{T}}$ with a vector **a**, in two ways—on the left, and on the right:

$$\left(\mathbf{a} \cdot \overleftrightarrow{\mathbf{T}}\right)_j = \sum_{i=x,y,z} a_i T_{ij}, \quad \left(\overleftrightarrow{\mathbf{T}} \cdot \mathbf{a}\right)_j = \sum_{i=x,y,z} T_{ji} a_i. \tag{8.18}$$

The resulting object, which has one remaining index, is itself a vector. In particular, the divergence of $\overleftrightarrow{\mathbf{T}}$ has as its jth component

$$\begin{aligned}\left(\nabla \cdot \overleftrightarrow{\mathbf{T}}\right)_j = \epsilon_0 \left[(\nabla \cdot \mathbf{E})E_j + (\mathbf{E} \cdot \nabla)E_j - \frac{1}{2}\nabla_j E^2\right] \\ + \frac{1}{\mu_0}\left[(\nabla \cdot \mathbf{B})B_j + (\mathbf{B} \cdot \nabla)B_j - \frac{1}{2}\nabla_j B^2\right].\end{aligned}$$

Thus the force per unit volume (Eq. 8.16) can be written in the much tidier form

$$\mathbf{f} = \nabla \cdot \overleftrightarrow{\mathbf{T}} - \epsilon_0 \mu_0 \frac{\partial \mathbf{S}}{\partial t}, \tag{8.19}$$

where **S** is the Poynting vector (Eq. 8.10).

The *total* electromagnetic force on the charges in $\mathcal{V}$ (Eq. 8.13) is

$$\mathbf{F} = \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} - \epsilon_0\mu_0 \frac{d}{dt}\int_{\mathcal{V}} \mathbf{S}\, d\tau. \tag{8.20}$$

(I used the divergence theorem to convert the first term to a surface integral.) In the *static* case the second term drops out, and the electromagnetic force on the charge configuration can be expressed entirely in terms of the stress tensor at the boundary:

$$\mathbf{F} = \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} \quad \text{(static)}. \tag{8.21}$$

Physically, $\overleftrightarrow{\mathbf{T}}$ is the force per unit area (or **stress**) acting on the surface. More precisely, T_{ij} is the force (per unit area) in the ith direction acting on an element of surface oriented in the jth direction—"diagonal" elements (T_{xx}, T_{yy}, T_{zz}) represent *pressures*, and "off-diagonal" elements (T_{xy}, T_{xz}, etc.) are *shears*.

Example 8.2. Determine the net force on the "northern" hemisphere of a uniformly charged solid sphere of radius R and charge Q (the same as Prob. 2.47, only this time we'll use the Maxwell stress tensor and Eq. 8.21).

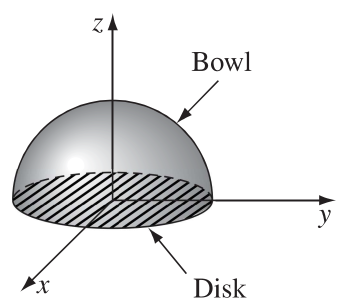

FIGURE 8.4

Solution

The boundary surface consists of two parts—a hemispherical "bowl" at radius R, and a circular disk at $\theta = \pi/2$ (Fig. 8.4). For the bowl,

$$d\mathbf{a} = R^2 \sin\theta\, d\theta\, d\phi\, \hat{\mathbf{r}}$$

and

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\,\hat{\mathbf{r}}.$$

In Cartesian components,

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}},$$

so

$$T_{zx} = \epsilon_0 E_z E_x = \epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 \sin\theta\cos\theta\cos\phi,$$

$$T_{zy} = \epsilon_0 E_z E_y = \epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 \sin\theta\cos\theta\sin\phi,$$

$$T_{zz} = \frac{\epsilon_0}{2}\left(E_z^2 - E_x^2 - E_y^2\right) = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^2}\right)^2 (\cos^2\theta - \sin^2\theta). \tag{8.22}$$

The net force is obviously in the z-direction, so it suffices to calculate

$$\left(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}\right)_z = T_{zx}\,da_x + T_{zy}\,da_y + T_{zz}\,da_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R}\right)^2 \sin\theta\cos\theta\,d\theta\,d\phi.$$

The force on the "bowl" is therefore

$$F_{\text{bowl}} = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R}\right)^2 2\pi\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{8R^2}. \tag{8.23}$$

Meanwhile, for the equatorial disk,

$$d\mathbf{a} = -r\,dr\,d\phi\,\hat{\mathbf{z}}, \tag{8.24}$$

and (since we are now *inside* the sphere)

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\mathbf{r} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}r(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}).$$

Thus

$$T_{zz} = \frac{\epsilon_0}{2}\left(E_z^2 - E_x^2 - E_y^2\right) = -\frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 r^2,$$

and hence

$$\left(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}\right)_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 r^3\,dr\,d\phi.$$

The force on the disk is therefore

$$F_{\text{disk}} = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 2\pi\int_0^R r^3 dr = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{16R^2}. \tag{8.25}$$

Combining Eqs. 8.23 and 8.25, I conclude that the net force on the northern hemisphere is

$$F = \frac{1}{4\pi\epsilon_0}\frac{3Q^2}{16R^2}. \tag{8.26}$$

Incidentally, in applying Eq. 8.21, *any* volume that encloses all of the charge in question (and no *other* charge) will do the job. For example, in the present case we could use the whole region $z > 0$. In that case the boundary surface consists of the entire xy plane (plus a hemisphere at $r = \infty$, but $E = 0$ out there, so it contributes nothing). In place of the "bowl," we now have the outer portion of the plane ($r > R$). Here

$$T_{zz} = -\frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 \frac{1}{r^4}$$

(Eq. 8.22 with $\theta = \pi/2$ and $R \to r$), and $d\mathbf{a}$ is given by Eq. 8.24, so

$$\left(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}\right)_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 \frac{1}{r^3}\, dr\, d\phi,$$

and the contribution from the plane for $r > R$ is

$$\frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 2\pi \int_R^\infty \frac{1}{r^3}\, dr = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{8R^2},$$

the same as for the bowl (Eq. 8.23).

I hope you didn't get too bogged down in the details of Ex. 8.2. If so, take a moment to appreciate what happened. We were calculating the force on a solid object, but instead of doing a *volume* integral, as you might expect, Eq. 8.21 allowed us to set it up as a *surface* integral; somehow the stress tensor sniffs out what is going on inside.

! **Problem 8.3** Calculate the force of magnetic attraction between the northern and southern hemispheres of a uniformly charged spinning spherical shell, with radius R, angular velocity ω, and surface charge density σ. [This is the same as Prob. 5.44, but this time use the Maxwell stress tensor and Eq. 8.21.]

Problem 8.4

(a) Consider two equal point charges q, separated by a distance $2a$. Construct the plane equidistant from the two charges. By integrating Maxwell's stress tensor over this plane, determine the force of one charge on the other.

(b) Do the same for charges that are opposite in sign.

8.2.3 ■ Conservation of Momentum

According to Newton's second law, the force on an object is equal to the rate of change of its momentum:

$$\mathbf{F} = \frac{d\mathbf{p}_{\text{mech}}}{dt}.$$

Equation 8.20 can therefore be written in the form[5]

$$\frac{d\mathbf{p}_{\text{mech}}}{dt} = -\epsilon_0\mu_0 \frac{d}{dt}\int_{\mathcal{V}} \mathbf{S}\,d\tau + \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}, \tag{8.27}$$

where $\mathbf{p}_{\text{mech}}$ is the (mechanical) momentum of the particles in volume $\mathcal{V}$. This expression is similar in structure to Poynting's theorem (Eq. 8.11), and it invites an analogous interpretation: The first integral represents *momentum stored in the fields*:

$$\mathbf{p} = \mu_0\epsilon_0 \int_{\mathcal{V}} \mathbf{S}\,d\tau, \tag{8.28}$$

while the second integral is the *momentum per unit time flowing in through the surface*.

Equation 8.27 is the statement of *conservation of momentum* in electrodynamics: If the mechanical momentum increases, either the field momentum *de*creases, or else the fields are carrying momentum into the volume through the surface. The momentum *density* in the fields is evidently

$$\boxed{\mathbf{g} = \mu_0\epsilon_0\mathbf{S} = \epsilon_0(\mathbf{E}\times\mathbf{B}),} \tag{8.29}$$

and the momentum flux transported by the fields is $-\overleftrightarrow{\mathbf{T}}$ (specifically, $-\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}$ is the electromagnetic momentum per unit time passing through the area $d\mathbf{a}$).

If the mechanical momentum in $\mathcal{V}$ is not changing (for example, if we are talking about a region of empty space), then

$$\int \frac{\partial \mathbf{g}}{\partial t}\,d\tau = \oint \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} = \int \nabla \cdot \overleftrightarrow{\mathbf{T}}\,d\tau,$$

and hence

$$\frac{\partial \mathbf{g}}{\partial t} = \nabla \cdot \overleftrightarrow{\mathbf{T}}. \tag{8.30}$$

This is the "continuity equation" for electromagnetic momentum, with $\mathbf{g}$ (momentum density) in the role of ρ (charge density) and $-\overleftrightarrow{\mathbf{T}}$ playing the part of $\mathbf{J}$; it expresses the local conservation of field momentum. But in general (when there *are* charges around) the field momentum by itself, and the mechanical momentum by itself, are *not* conserved—charges and fields exchange momentum, and only the *total* is conserved.

Notice that the Poynting vector has appeared in two quite different roles: $\mathbf{S}$ itself is the energy per unit area, per unit time, transported by the electromagnetic fields, while $\mu_0\epsilon_0\mathbf{S}$ is the momentum per unit volume stored in those fields.[6]

[5]Let's assume the only forces acting are electromagnetic. You can include other forces if you like—both here and in the discussion of energy conservation—but they are just a distraction from the essential story.

[6]This is no coincidence—see R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Reading, Mass.: Addison-Wesley, 1964), Vol. II, Section 27-6.

Similarly, $\overleftrightarrow{\mathbf{T}}$ plays a dual role: $\overleftrightarrow{\mathbf{T}}$ itself is the electromagnetic stress (force per unit area) acting on a surface, and $-\overleftrightarrow{\mathbf{T}}$ describes the flow of momentum (it is the momentum current density) carried by the fields.

Example 8.3. A long coaxial cable, of length l, consists of an inner conductor (radius a) and an outer conductor (radius b). It is connected to a battery at one end and a resistor at the other (Fig. 8.5). The inner conductor carries a uniform charge per unit length λ, and a steady current I to the right; the outer conductor has the opposite charge and current. What is the electromagnetic momentum stored in the fields?

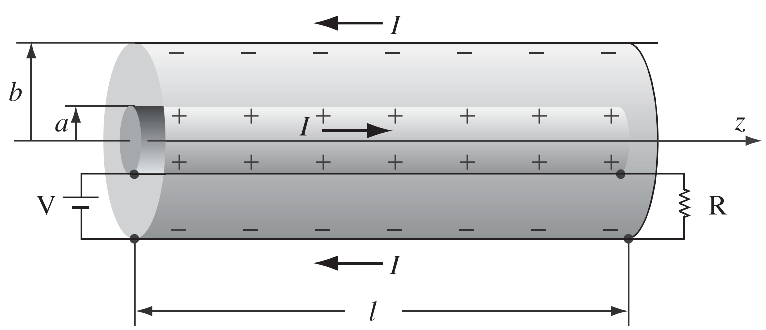

FIGURE 8.5

Solution
The fields are

$$\mathbf{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\,\hat{\mathbf{s}}, \qquad \mathbf{B} = \frac{\mu_0}{2\pi}\frac{I}{s}\,\hat{\boldsymbol{\phi}}.$$

The Poynting vector is therefore

$$\mathbf{S} = \frac{\lambda I}{4\pi^2\epsilon_0 s^2}\,\hat{\mathbf{z}}.$$

So energy is flowing down the line, from the battery to the resistor. In fact, the power transported is

$$P = \int \mathbf{S}\cdot d\mathbf{a} = \frac{\lambda I}{4\pi^2\epsilon_0}\int_a^b \frac{1}{s^2}2\pi s\,ds = \frac{\lambda I}{2\pi\epsilon_0}\ln(b/a) = IV,$$

as it should be.

The *momentum* in the fields is

$$\mathbf{p} = \mu_0\epsilon_0\int \mathbf{S}\,d\tau = \frac{\mu_0\lambda I}{4\pi^2}\,\hat{\mathbf{z}}\int_a^b \frac{1}{s^2}l2\pi s\,ds = \frac{\mu_0\lambda Il}{2\pi}\ln(b/a)\,\hat{\mathbf{z}} = \frac{IVl}{c^2}\,\hat{\mathbf{z}}.$$

This is an astonishing result. The cable is not moving, **E** and **B** are static, and yet we are asked to believe that there is momentum in the fields. If something tells you this cannot be the whole story, you have sound intuitions. But the resolution of this paradox will have to await Chapter 12 (Ex. 12.12).

Suppose now that we turn up the resistance, so the current decreases. The changing magnetic field will induce an electric field (Eq. 7.20):

$$\mathbf{E} = \left[\frac{\mu_0}{2\pi} \frac{dI}{dt} \ln s + K \right] \hat{\mathbf{z}}.$$

This field exerts a force on $\pm\lambda$:

$$\mathbf{F} = \lambda l \left[\frac{\mu_0}{2\pi} \frac{dI}{dt} \ln a + K \right] \hat{\mathbf{z}} - \lambda l \left[\frac{\mu_0}{2\pi} \frac{dI}{dt} \ln b + K \right] \hat{\mathbf{z}} = -\frac{\mu_0 \lambda l}{2\pi} \frac{dI}{dt} \ln(b/a)\, \hat{\mathbf{z}}.$$

The total momentum imparted to the cable, as the current drops from I to 0, is therefore

$$\mathbf{p}_{\text{mech}} = \int \mathbf{F}\, dt = \frac{\mu_0 \lambda I l}{2\pi} \ln(b/a)\, \hat{\mathbf{z}},$$

which is precisely the momentum originally stored in the fields.

Problem 8.5 Imagine two parallel infinite sheets, carrying uniform surface charge $+\sigma$ (on the sheet at $z = d$) and $-\sigma$ (at $z = 0$). They are moving in the y direction at constant speed v (as in Problem 5.17).

(a) What is the electromagnetic momentum in a region of area A?

(b) Now suppose the top sheet moves slowly down (speed u) until it reaches the bottom sheet, so the fields disappear. By calculating the total force on the charge ($q = \sigma A$), show that the impulse delivered to the sheet is equal to the momentum originally stored in the fields. [*Hint:* As the upper plate passes by, the magnetic field drops to zero, inducing an electric field that delivers an impulse to the lower plate.]

Problem 8.6 A charged parallel-plate capacitor (with uniform electric field $\mathbf{E} = E\,\hat{\mathbf{z}}$) is placed in a uniform magnetic field $\mathbf{B} = B\,\hat{\mathbf{x}}$, as shown in Fig. 8.6.

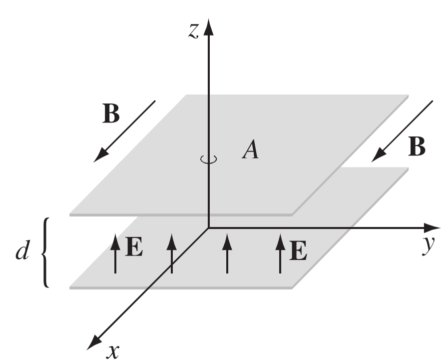

FIGURE 8.6

(a) Find the electromagnetic momentum in the space between the plates.

(b) Now a resistive wire is connected between the plates, along the z axis, so that the capacitor slowly discharges. The current through the wire will experience a magnetic force; what is the total impulse delivered to the system, during the discharge?[7]

Problem 8.7 Consider an infinite parallel-plate capacitor, with the lower plate (at $z = -d/2$) carrying surface charge density $-\sigma$, and the upper plate (at $z = +d/2$) carrying charge density $+\sigma$.

(a) Determine all nine elements of the stress tensor, in the region between the plates. Display your answer as a 3×3 matrix:

$$\begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

(b) Use Eq. 8.21 to determine the electromagnetic force per unit area on the top plate. Compare Eq. 2.51.

(c) What is the electromagnetic momentum per unit area, per unit time, crossing the xy plane (or any other plane parallel to that one, between the plates)?

(d) Of course, there must be *mechanical* forces holding the plates apart—perhaps the capacitor is filled with insulating material under pressure. Suppose we suddenly *remove* the insulator; the momentum flux (c) is now absorbed by the plates, and they begin to move. Find the momentum per unit time delivered to the top plate (which is to say, the force acting on it) and compare your answer to (b). [*Note:* This is not an *additional* force, but rather an alternative way of calculating the *same* force—in (b) we got it from the force law, and in (d) we do it by conservation of momentum.]

8.2.4 ■ Angular Momentum

By now, the electromagnetic fields (which started out as mediators of forces between charges) have taken on a life of their own. They carry *energy* (Eq. 8.5)

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right), \tag{8.31}$$

and *momentum* (Eq. 8.29)

$$\mathbf{g} = \epsilon_0(\mathbf{E} \times \mathbf{B}), \tag{8.32}$$

[7]There is *much* more to be said about this problem, so don't get too excited if your answers to (a) and (b) appear to be consistent. See D. Babson, et al., *Am. J. Phys.* **77**, 826 (2009).

and, for that matter, *angular* momentum:

$$\boldsymbol{\ell} = \mathbf{r} \times \mathbf{g} = \epsilon_0 \left[\mathbf{r} \times (\mathbf{E} \times \mathbf{B})\right]. \tag{8.33}$$

Even perfectly *static* fields can harbor momentum and angular momentum, as long as $\mathbf{E} \times \mathbf{B}$ is nonzero, and it is only when these field contributions are included that the conservation laws are sustained.

Example 8.4. Imagine a very long solenoid with radius R, n turns per unit length, and current I. Coaxial with the solenoid are two long cylindrical (nonconducting) shells of length l—one, *inside* the solenoid at radius a, carries a charge $+Q$, uniformly distributed over its surface; the other, *outside* the solenoid at radius b, carries charge $-Q$ (see Fig. 8.7; l is supposed to be much greater than b). When the current in the solenoid is gradually reduced, the cylinders begin to rotate, as we found in Ex. 7.8. *Question*: Where does the angular momentum come from?[8]

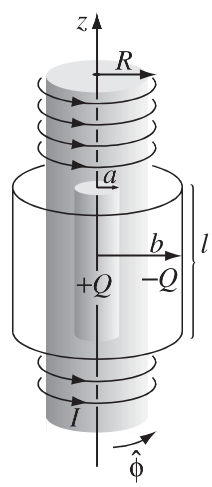

FIGURE 8.7

Solution

It was initially stored in the fields. Before the current was switched off, there was an electric field,

$$\mathbf{E} = \frac{Q}{2\pi\epsilon_0 l}\frac{1}{s}\,\hat{\mathbf{s}} \quad (a < s < b),$$

in the region between the cylinders, and a magnetic field,

$$\mathbf{B} = \mu_0 n I\,\hat{\mathbf{z}} \quad (s < R),$$

[8]This is a variation on the "Feynman disk paradox" (R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures*, vol 2, pp. 17-5 (Reading, Mass.: Addison-Wesley, 1964) suggested by F. L. Boos, Jr. (*Am. J. Phys.* **52**, 756 (1984)). A similar model was proposed earlier by R. H. Romer (*Am. J. Phys.* **34**, 772 (1966)). For further references, see T.-C. E. Ma, *Am. J. Phys.* **54**, 949 (1986).

inside the solenoid. The momentum density (Eq. 8.29) was therefore

$$\mathbf{g} = -\frac{\mu_0 n I Q}{2\pi l s}\,\hat{\boldsymbol{\phi}},$$

in the region $a < s < R$. The z component of the *angular* momentum density was

$$(\mathbf{r} \times \mathbf{g})_z = -\frac{\mu_0 n I Q}{2\pi l},$$

which is *constant* (independent of s). To get the *total* angular momentum in the fields, we simply multiply by the volume, $\pi(R^2 - a^2)l$:[9]

$$\mathbf{L} = -\frac{1}{2}\mu_0 n I Q(R^2 - a^2)\,\hat{\mathbf{z}}. \tag{8.34}$$

When the current is turned off, the changing magnetic field induces a circumferential electric field, given by Faraday's law:

$$\mathbf{E} = \begin{cases} -\frac{1}{2}\mu_0 n \frac{dI}{dt}\frac{R^2}{s}\,\hat{\boldsymbol{\phi}}, & (s > R), \\ -\frac{1}{2}\mu_0 n \frac{dI}{dt} s\,\hat{\boldsymbol{\phi}}, & (s < R). \end{cases}$$

Thus the torque on the outer cylinder is

$$\mathbf{N}_b = \mathbf{r} \times (-Q\mathbf{E}) = \frac{1}{2}\mu_0 n Q R^2 \frac{dI}{dt}\hat{\mathbf{z}},$$

and it picks up an angular momentum

$$\mathbf{L}_b = \frac{1}{2}\mu_0 n Q R^2\,\hat{\mathbf{z}} \int_I^0 \frac{dI}{dt}dt = -\frac{1}{2}\mu_0 n I Q R^2\,\hat{\mathbf{z}}.$$

Similarly, the torque on the inner cylinder is

$$\mathbf{N}_a = -\frac{1}{2}\mu_0 n Q a^2 \frac{dI}{dt}\,\hat{\mathbf{z}},$$

and its angular momentum increase is

$$\mathbf{L}_a = \frac{1}{2}\mu_0 n I Q a^2\,\hat{\mathbf{z}}.$$

So it all works out: $\mathbf{L}_{\text{em}} = \mathbf{L}_a + \mathbf{L}_b$. The angular momentum *lost* by the fields is precisely equal to the angular momentum *gained* by the cylinders, and the *total* angular momentum (fields plus matter) is conserved.

---

[9] The radial component integrates to zero, by symmetry.

Problem 8.8 In Ex. 8.4, suppose that instead of turning off the *magnetic* field (by reducing I) we turn off the *electric* field, by connecting a weakly[10] conducting radial spoke between the cylinders. (We'll have to cut a slot in the solenoid, so the cylinders can still rotate freely.) From the magnetic force on the current in the spoke, determine the total angular momentum delivered to the cylinders, as they discharge (they are now rigidly connected, so they rotate together). Compare the initial angular momentum stored in the fields (Eq. 8.34). (Notice that the *mechanism* by which angular momentum is transferred from the fields to the cylinders is entirely different in the two cases: in Ex. 8.4 it was Faraday's law, but here it is the Lorentz force law.)

Problem 8.9 Two concentric spherical shells carry uniformly distributed charges $+Q$ (at radius a) and $-Q$ (at radius $b > a$). They are immersed in a uniform magnetic field $\mathbf{B} = B_0\,\hat{\mathbf{z}}$.

(a) Find the angular momentum of the fields (with respect to the center).

(b) Now the magnetic field is gradually turned off. Find the torque on each sphere, and the resulting angular momentum of the system.

! **Problem 8.10**[11] Imagine an iron sphere of radius R that carries a charge Q and a uniform magnetization $\mathbf{M} = M\hat{\mathbf{z}}$. The sphere is initially at rest.

(a) Compute the angular momentum stored in the electromagnetic fields.

(b) Suppose the sphere is gradually (and uniformly) demagnetized (perhaps by heating it up past the Curie point). Use Faraday's law to determine the induced electric field, find the torque this field exerts on the sphere, and calculate the total angular momentum imparted to the sphere in the course of the demagnetization.

(c) Suppose instead of *demagnetizing* the sphere we *discharge* it, by connecting a grounding wire to the north pole. Assume the current flows over the surface in such a way that the charge density remains uniform. Use the Lorentz force law to determine the torque on the sphere, and calculate the total angular momentum imparted to the sphere in the course of the discharge. (The magnetic field is discontinuous at the surface ... does this matter?) [*Answer:* $\frac{2}{9}\mu_0 M Q R^2$]

8.3 ■ MAGNETIC FORCES DO NO WORK[12]

This is perhaps a good place to revisit the old paradox that magnetic forces do no work (Eq. 5.11). What about that magnetic crane lifting the carcass of a junked car? *Somebody* is doing work on the car, and if it's not the magnetic field, who

[10]In Ex. 8.4 we turned the current off slowly, to keep things quasistatic; here we reduce the electric field slowly to keep the displacement current negligible.

[11]This version of the Feynman disk paradox was proposed by N. L. Sharma (*Am. J. Phys.* **56**, 420 (1988)); similar models were analyzed by E. M. Pugh and G. E. Pugh, *Am. J. Phys.* **35**, 153 (1967) and by R. H. Romer, *Am. J. Phys.* **35**, 445 (1967).

[12]This section can be skipped without loss of continuity. I include it for those readers who are disturbed by the notion that magnetic forces do no work.

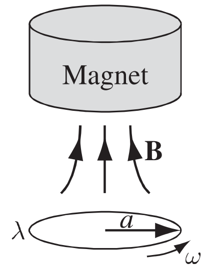

FIGURE 8.8

is it? The car is ferromagnetic; in the presence of the magnetic field, it contains a lot of microscopic magnetic dipoles (spinning electrons, actually), all lined up. The resulting magnetization is equivalent to a bound current running around the surface, so let's model the car as a circular current loop—in fact, let's make it an insulating ring of line charge λ rotating at angular velocity ω (Fig. 8.8).

The upward magnetic force on the loop is (Eq. 6.2)

$$F = 2\pi I a B_s, \tag{8.35}$$

where B_s is the radial component of the magnet's field,[13] and $I = \lambda\omega a$. If the ring rises a distance dz (while the magnet itself stays put), the work done on it is

$$dW = 2\pi a^2 \lambda\omega B_s\, dz. \tag{8.36}$$

This increases the potential energy of the ring. Who did the work? Naively, it appears that the magnetic field is responsible, but we have already learned (Ex. 5.3) that this is not the case—as the ring rises, the magnetic force is perpendicular to the *net* velocity of the charges in the ring, so it does *no* work on them.

At the same time, however, a motional emf is induced in the ring, which opposes the flow of charge, and hence reduces its angular velocity:

$$\mathcal{E} = -\frac{d\Phi}{dt}.$$

Here $d\Phi$ is the flux through the "ribbon" joining the ring at time t to the ring at time $t + dt$ (Fig. 8.9):

$$d\Phi = B_s\, 2\pi a\, dz.$$

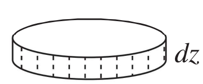

FIGURE 8.9

[13]Note that the field has to be *nonuniform*, or it won't lift the car at all.

Now

$$\mathcal{E} = \oint \mathbf{f} \cdot d\mathbf{l} = f(2\pi a),$$

where $\mathbf{f}$ is the force per unit charge. So

$$f = -B_s \frac{dz}{dt}, \tag{8.37}$$

the force on a segment of length dl is $f\lambda\, dl$, the torque on the ring is

$$N = a\left(-B_s \frac{dz}{dt}\right)\lambda(2\pi a),$$

and the work done (slowing the rotation) is $N\, d\phi = N\omega\, dt$, or

$$dW = -2\pi a^2 \lambda\omega B_s\, dz. \tag{8.38}$$

The ring slows down, and the rotational energy it loses (Eq. 8.38) is precisely equal to the potential energy it gains (Eq. 8.36). All the magnetic field did was convert energy from one form to another. If you'll permit some sloppy language, the work done by the vertical component of the magnetic force (Eq. 8.35) is equal and opposite to the work done by its horizontal component (Eq. 8.37).[14]

What about the magnet? Is it completely passive in this process? Suppose we model it as a big circular loop (radius b), resting on a table and carrying a current I_b; the "junk car" is a relatively small current loop (radius a), on the floor directly below, carrying a current I_a (Fig. 8.10). This time, just for a change, let's assume both currents are constant (we'll include a regulated power supply in each loop[15]). Parallel currents attract, and we propose to lift the small loop off the floor, keeping careful track of the work done and the agency responsible.

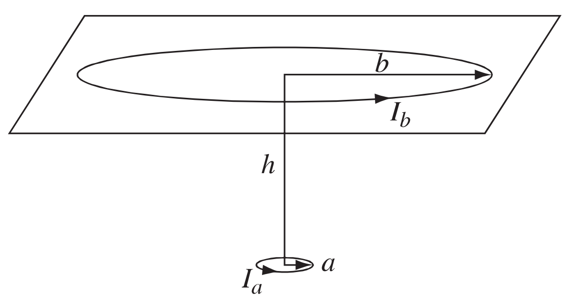

FIGURE 8.10

[14]This argument is essentially the same as the one in Ex. 5.3, except that in this case I told the story in terms of motional emf, instead of the Lorentz force law. But after all, the flux rule is a *consequence* of the Lorentz force law.

[15]The lower loop could be a single spinning electron, in which case quantum mechanics fixes its angular momentum at $\hbar/2$. It might appear that this sustains the current, with no need for a power supply. I will return to this point, but for now let's just keep quantum mechanics out of it.

Let's start by adjusting the currents so the small ring just "floats," a distance h below the table, with the magnetic force exactly balancing the weight ($m_a g$) of the little ring. I'll let you calculate the magnetic force (Prob. 8.11):

$$F_{\text{mag}} = \frac{3\pi}{2}\mu_0 I_a I_b \frac{a^2 b^2 h}{(b^2+h^2)^{5/2}} = m_a g. \tag{8.39}$$

Now the loop rises an infinitesimal distance dz; the work done is equal to the gain in its potential energy

$$dW_g = m_a g\, dz = \frac{3\pi}{2}\mu_0 I_a I_b \frac{a^2 b^2 h}{(b^2+h^2)^{5/2}}\, dz. \tag{8.40}$$

Who did it? The magnetic field? *No!* The work was done by the power supply that sustains the current in loop a (Ex. 5.3). As the loop rises, a motional emf is induced in it. The flux through the loop is

$$\Phi_a = M I_b,$$

where M is the mutual inductance of the two loops:

$$M = \frac{\pi\mu_0}{2}\frac{a^2 b^2}{(b^2+h^2)^{3/2}}$$

(Prob. 7.22). The emf is

$$\begin{aligned}\mathcal{E}_a &= -\frac{d\Phi_a}{dt} = -I_b\frac{dM}{dt} = -I_b\frac{dM}{dh}\frac{dh}{dt}\\ &= -I_b\left(-\frac{3}{2}\right)\frac{\pi\mu_0}{2}\frac{a^2b^2}{(b^2+h^2)^{5/2}}2h\frac{(-dz)}{dt}.\end{aligned}$$

The work done by the power supply (fighting against this motional emf) is

$$dW_a = -\mathcal{E}_a I_a\, dt = \frac{3\pi}{2}\mu_0 I_a I_b \frac{a^2 b^2 h}{(b^2+h^2)^{5/2}}\, dz \tag{8.41}$$

—same as the work done in lifting the loop (Eq. 8.40).

Meanwhile, however, a *Faraday* emf is induced in the *upper* loop, due to the changing flux from the lower loop:

$$\Phi_b = M I_a \;\Rightarrow\; \mathcal{E}_b = -I_a\frac{dM}{dt},$$

and the work done by the power supply in ring b (to sustain the current I_b) is

$$dW_b = -\mathcal{E}_b I_b\, dt = \frac{3\pi}{2}\mu_0 I_a I_b \frac{a^2 b^2 h}{(b^2+h^2)^{5/2}}\, dz, \tag{8.42}$$

exactly the same as dW_a. That's embarrassing—the power supplies have done *twice* as much work as was necessary to lift the junk car! Where did the "wasted" energy go? *Answer:* It increased the energy stored in the fields. The energy in a system of two current-carrying loops is (see Prob. 8.12)

$$U = \frac{1}{2}L_a I_a^2 + \frac{1}{2}L_b I_b^2 + M I_a I_b, \tag{8.43}$$

so

$$dU = I_a I_b \frac{dM}{dt} dt = dW_b.$$

Remarkably, all four energy increments are the same. If we care to apportion things this way, the power supply in loop *a* contributes the energy necessary to lift the lower ring, while the power supply in loop *b* provides the extra energy for the fields. If all we're interested in is the work done to raise the ring, we can ignore the upper loop (and the energy in the fields) altogether.

In both these models, the magnet itself was stationary. That's like lifting a paper clip by holding a magnet over it. But in the case of the magnetic crane, the car stays in contact with the magnet, which is attached to a cable that lifts the whole works. As a model, we might stick the upper loop in a big box, the lower loop in a little box, and crank up the currents so the force of attraction is much greater than $m_a g$; the two boxes snap together, and we attach a string to the upper box and pull up on it (Fig. 8.11).

The same old mechanism (Ex. 5.3) prevails: as the lower loop rises, the magnetic force tilts backwards; its vertical component lifts the loop, but its horizontal component opposes the current, and no net work is done. This time, however, the motional emf is perfectly balanced by the Faraday emf fighting to keep the current going—the flux through the lower loop is not changing. (If you like, the flux is *increasing* because the loop is moving upward, into a region of higher magnetic field, but it is *decreasing* because the magnetic field of the upper loop—at any give point in space—is decreasing as that loop moves up.) No power supply is needed to sustain the current (and for that matter, no power supply is required in the upper loop either, since the energy in the fields is not changing. Who did the work to lift the car? The person pulling up on the rope, obviously. The role of the magnetic field was merely to transmit this energy to the car, via the vertical component of the magnetic force. But the magnetic field itself (as always) did no work.

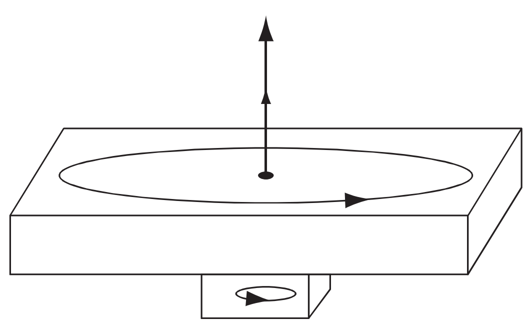

FIGURE 8.11

The fact that magnetic fields do no work follows directly from the Lorentz force law, so if you think you have discovered an exception, you're going to have to explain why that law is incorrect. For example, if magnetic monopoles exist, the force on a particle with electric charge q_e and magnetic charge q_m becomes (Prob. 7.38):

$$\mathbf{F} = q_e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + q_m\left(\mathbf{B} - \epsilon_0\mu_0\mathbf{v} \times \mathbf{E}\right). \tag{8.44}$$

In that case, magnetic fields *can* do work ... but *only on magnetic charges*. So unless your car is made of monopoles (I don't think so), this doesn't solve the problem.

A somewhat less radical possibility is that in addition to electric charges there exist permanent point magnetic dipoles (electrons?), whose dipole moment $\mathbf{m}$ is not associated with any electric current, but simply *is*. The Lorentz force law acquires an extra term

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \nabla(\mathbf{m} \cdot \mathbf{B}).$$

The magnetic field *can* do work on these "intrinsic" dipoles (which experience no motional or Faraday emf, since they enclose no flux). I don't know whether a consistent theory can be constructed in this way, but in any event it is *not* classical electrodynamics, which is predicated on Ampère's assumption that all magnetic phenomena are due to electric charges in motion, and point magnetic dipoles must be interpreted as the limits of tiny current loops.

Problem 8.11 Derive Eq. 8.39. [*Hint:* Treat the lower loop as a magnetic dipole.]

Problem 8.12 Derive Eq. 8.43. [*Hint:* Use the method of Section 7.2.4, building the two currents up from zero to their final values.]

More Problems on Chapter 8

Problem 8.13[16] A very long solenoid of radius a, with n turns per unit length, carries a current I_s. Coaxial with the solenoid, at radius $b \gg a$, is a circular ring of wire, with resistance R. When the current in the solenoid is (gradually) decreased, a current I_r is induced in the ring.

(a) Calculate I_r, in terms of dI_s/dt.

(b) The power ($I_r^2 R$) delivered to the ring must have come from the solenoid. Confirm this by calculating the Poynting vector just outside the solenoid (the *electric* field is due to the changing flux in the solenoid; the *magnetic* field is due to the current in the ring). Integrate over the entire surface of the solenoid, and check that you recover the correct total power.

[16] For extensive discussion, see M. A. Heald, *Am. J. Phys.* **56**, 540 (1988).

Problem 8.14 An infinitely long cylindrical tube, of radius a, moves at constant speed v along its axis. It carries a net charge per unit length λ, uniformly distributed over its surface. Surrounding it, at radius b, is another cylinder, moving with the same velocity but carrying the opposite charge ($-\lambda$). Find:

(a) The energy per unit length stored in the fields.

(b) The momentum per unit length in the fields.

(c) The energy per unit time transported by the fields across a plane perpendicular to the cylinders.

Problem 8.15 A point charge q is located at the center of a toroidal coil of rectangular cross section, inner radius a, outer radius $a + w$, and height h, which carries a total of N tightly-wound turns and current I.

(a) Find the electromagnetic momentum $\mathbf{p}$ of this configuration, assuming that w and h are both much less than a (so you can ignore the variation of the fields over the cross section).

(b) Now the current in the toroid is turned off, quickly enough that the point charge does not move appreciably as the magnetic field drops to zero. Show that the impulse imparted to q is equal to the momentum originally stored in the electromagnetic fields. [*Hint:* You might want to refer to Prob. 7.19.]

Problem 8.16[17] A sphere of radius R carries a uniform polarization $\mathbf{P}$ and a uniform magnetization $\mathbf{M}$ (not necessarily in the same direction). Find the electromagnetic momentum of this configuration. [*Answer:* $(4/9)\pi\mu_0 R^3(\mathbf{M} \times \mathbf{P})$]

Problem 8.17[18] Picture the electron as a uniformly charged spherical shell, with charge e and radius R, spinning at angular velocity ω.

(a) Calculate the total energy contained in the electromagnetic fields.

(b) Calculate the total angular momentum contained in the fields.

(c) According to the Einstein formula ($E = mc^2$), the energy in the fields should contribute to the mass of the electron. Lorentz and others speculated that the *entire* mass of the electron might be accounted for in this way: $U_{\text{em}} = m_e c^2$. Suppose, moreover, that the electron's spin angular momentum is entirely attributable to the electromagnetic fields: $L_{\text{em}} = \hbar/2$. On these two assumptions, determine the radius and angular velocity of the electron. What is their product, ωR? Does this classical model make sense?

Problem 8.18 Work out the formulas for u, $\mathbf{S}$, $\mathbf{g}$, and $\overleftrightarrow{\mathbf{T}}$ in the presence of magnetic charge. [*Hint:* Start with the generalized Maxwell equations (7.44) and Lorentz force law (Eq. 8.44), and follow the derivations in Sections 8.1.2, 8.2.2, and 8.2.3.]

[17] For an interesting discussion and references, see R. H. Romer, *Am. J. Phys.* **63**, 777 (1995).
[18] See J. Higbie, *Am. J. Phys.* **56**, 378 (1988).

! **Problem 8.19**[19] Suppose you had an electric charge q_e and a magnetic monopole q_m. The field of the electric charge is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q_e}{\mathscr{r}^2}\hat{\mathscr{r}}$$

(of course), and the field of the magnetic monopole is

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q_m}{\mathscr{r}^2}\hat{\mathscr{r}}.$$

Find the total angular momentum stored in the fields, if the two charges are separated by a distance d. [*Answer:* $(\mu_0/4\pi)q_e q_m$.][20]

Problem 8.20 Consider an ideal stationary magnetic dipole $\mathbf{m}$ in a static electric field $\mathbf{E}$. Show that the fields carry momentum

$$\mathbf{p} = -\epsilon_0\mu_0(\mathbf{m}\times\mathbf{E}). \tag{8.45}$$

[*Hint:* There are several ways to do this. The simplest method is to start with $\mathbf{p} = \epsilon_0\int(\mathbf{E}\times\mathbf{B})\,d\tau$, write $\mathbf{E} = -\nabla V$, and use integration by parts to show that

$$\mathbf{p} = \epsilon_0\mu_0\int V\mathbf{J}\,d\tau.$$

So far, this is valid for *any* localized static configuration. For a current confined to an infinitesimal neighborhood of the origin we can approximate $V(\mathbf{r}) \approx V(\mathbf{0}) - \mathbf{E}(\mathbf{0})\cdot\mathbf{r}$. Treat the dipole as a current loop, and use Eqs. 5.82 and 1.108.][21]

Problem 8.21 Because the cylinders in Ex. 8.4 are left rotating (at angular velocities ω_a and ω_b, say), there is actually a residual magnetic field, and hence angular momentum in the fields, even after the current in the solenoid has been extinguished. If the cylinders are heavy, this correction will be negligible, but it is interesting to do the problem *without* making that assumption.[22]

(a) Calculate (in terms of ω_a and ω_b) the final angular momentum in the fields. [Define $\boldsymbol{\omega} = \omega\,\hat{\mathbf{z}}$, so ω_a and ω_b could be positive or negative.]

(b) As the cylinders begin to rotate, their changing magnetic field induces an extra azimuthal electric field, which, in turn, will make an additional contribution to

[19] This system is known as **Thomson's dipole**. See I. Adawi, *Am. J. Phys.* **44**, 762 (1976) and *Phys. Rev.* **D31**, 3301 (1985), and K. R. Brownstein, *Am. J. Phys.* **57**, 420 (1989), for discussion and references.

[20] Note that this result is *independent of the separation distance* d! It points from q_e toward q_m. In quantum mechanics, angular momentum comes in half-integer multiples of $\hbar$, so this result suggests that if magnetic monopoles exist, electric and magnetic charge must be quantized: $\mu_0 q_e q_m/4\pi = n\hbar/2$, for $n = 1, 2, 3, \ldots$, an idea first proposed by Dirac in 1931. If even *one* monopole is lurking somewhere in the universe, this would "explain" why electric charge comes in discrete units. (However, see D. Singleton, *Am. J. Phys.* **66**, 697 (1998) for a cautionary note.)

[21] As it stands, Eq. 8.45 is valid only for *ideal* dipoles. But $\mathbf{g}$ is linear in $\mathbf{B}$, and therefore, if $\mathbf{E}$ is held fixed, obeys the superposition principle: For a *collection* of magnetic dipoles, the total momentum is the (vector) sum of the momenta for each one separately. In particular, if $\mathbf{E}$ is *uniform* over a localized steady current distribution, then Eq. 8.45 is valid for the whole thing, only now $\mathbf{m}$ is the *total* magnetic dipole moment.

[22] This problem was suggested by Paul DeYoung.

the torques. Find the resulting extra angular momentum, and compare it with your result in (a). [*Answer:* $-\mu_0 Q^2 \omega_b (b^2 - a^2)/4\pi l\,\hat{\mathbf{z}}$]

Problem 8.22[23] A point charge q is a distance $a > R$ from the axis of an infinite solenoid (radius R, n turns per unit length, current I). Find the linear momentum and the angular momentum (with respect to the origin) in the fields. (Put q on the x axis, with the solenoid along z; treat the solenoid as a nonconductor, so you don't need to worry about induced charges on its surface.) [*Answer:* $\mathbf{p} = (\mu_0 q n I R^2/2a)\,\hat{\mathbf{y}}$; $\mathbf{L} = \mathbf{0}$]

Problem 8.23

(a) Carry through the argument in Sect. 8.1.2, starting with Eq. 8.6, but using $\mathbf{J}_f$ in place of $\mathbf{J}$. Show that the Poynting vector becomes

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}, \tag{8.46}$$

and the rate of change of the energy density in the fields is

$$\frac{\partial u}{\partial t} = \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}.$$

For *linear* media, show that[24]

$$u = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}). \tag{8.47}$$

(b) In the same spirit, reproduce the argument in Sect. 8.2.2, starting with Eq. 8.15, with ρ_f and $\mathbf{J}_f$ in place of ρ and $\mathbf{J}$. Don't bother to construct the Maxwell stress tensor, but do show that the momentum density is[25]

$$\mathbf{g} = \mathbf{D} \times \mathbf{B}. \tag{8.48}$$

Problem 8.24

A circular disk of radius R and mass M carries n point charges (q), attached at regular intervals around its rim. At time $t = 0$ the disk lies in the xy plane, with its center at the origin, and is rotating about the z axis with angular velocity ω_0, when it is released. The disk is immersed in a (time-independent) external magnetic field

$$\mathbf{B}(s, z) = k(-s\,\hat{\mathbf{s}} + 2z\,\hat{\mathbf{z}}),$$

where k is a constant.

(a) Find the position of the center if the ring, $z(t)$, and its angular velocity, $\omega(t)$, as functions of time. (Ignore gravity.)

(b) Describe the motion, and check that the total (kinetic) energy—translational plus rotational—is constant, confirming that the magnetic force does no work.[26]

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[23] See F. S. Johnson, B. L. Cragin, and R. R. Hodges, *Am. J. Phys.* **62**, 33 (1994), and B. Y.-K. Hu, *Eur. J. Phys.* **33**, 873 (2012), for discussion of this and related problems.

[24] Refer to Sect. 4.4.3 for the meaning of "energy" in this context.

[25] For over 100 years there has been a raging debate (still not completely resolved) as to whether the field momentum in polarizable/magnetizable media is Eq. 8.48 (Minkowski's candidate) or $\epsilon_0\mu_0$ ($\mathbf{E} \times \mathbf{H}$) (Abraham's). See D. J. Griffiths, *Am. J. Phys.* **80**, 7 (2012).

[26] This cute problem is due to K. T. McDonald, http://puhep1.princeton.edu/mcdonald/examles/disk.pdf (who draws a somewhat different conclusion).