Module 5: Two Dimensional Problems in Cartesian Coordinate System

5.1.1 INTRODUCTION

Plane Stress Problems

In many instances the stress situation is simpler than that illustrated in Figure 2.7. An example of practical interest is that of a thin plate which is being pulled by forces in the plane of the plate. Figure 5.1 shows a plate of constant thickness, t subjected to axial and shear stresses in the x and y directions only. The thickness is small compared to the other two dimensions of plate. These stresses are assumed to be uniformly distributed over the thickness t. The surface normal to the z-axis is stress free.

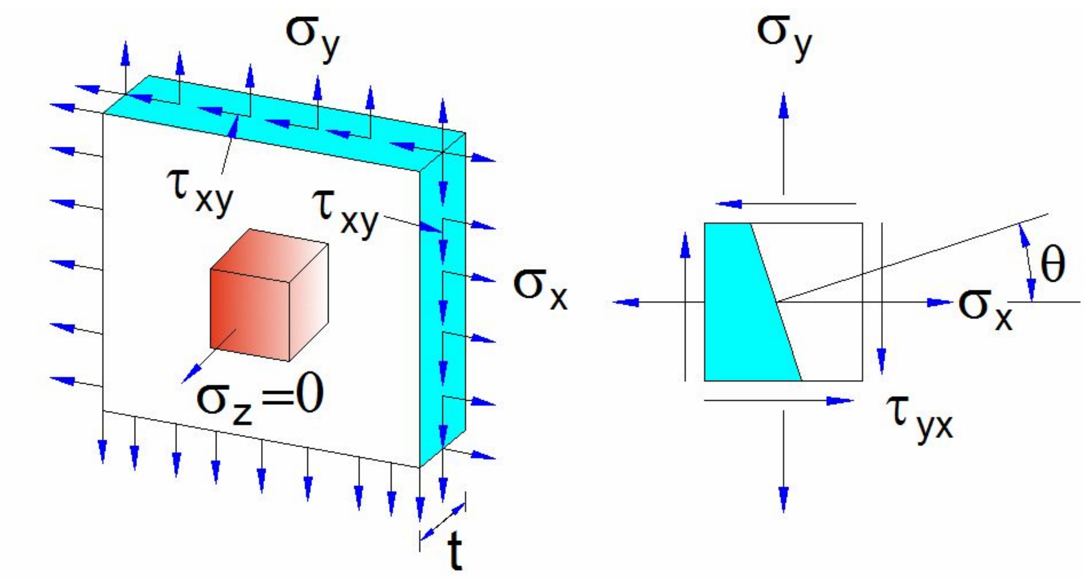


Figure 5.1 General case of plane stress

The state of stress at a given point will only depend upon the four stress components such as

$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix} \tag{5.0}$$

in which the stress components are functions of only x and y. This combination of stress components is called "plane stress" in the xy plane. The stress-strain relations for plane stress is given by

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right)$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right) \tag{5.1}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

and $\gamma_{xz} = \gamma_{yz} = 0, \varepsilon_z = -\frac{\nu}{E}\left(\sigma_x + \sigma_y\right)$

Compatibility Equation in terms of Stress Components (Plane stress case)

For two dimensional state of strain, the condition of compatibility (Eq. 3.21) is given by

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{5.1a}$$

Substituting Eq. 5.1 in Eq. 5.1a

$$\frac{\partial^2}{\partial y^2}(\sigma_x - \nu\sigma_y) + \frac{\partial^2}{\partial x^2}(\sigma_y - \nu\sigma_x) = 2(1+\nu)\frac{\partial^2 \tau_{xy}}{\partial x \partial y} \tag{5.1b}$$

Further equations of equilibrium are given by

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0 \tag{5.1c}$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + F_y = 0 \tag{5.1d}$$

Differentiate (5.1c) with respect to x and (5.1d) with respect to y and adding the two, we get

$$\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\left[\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right] \tag{5.1e}$$

Substituting Eq. (5.1e) in Eq. (5.1b), we get

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\sigma_x+\sigma_y\right)=-(1+\nu)\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right) \tag{5.2}$$

If the body forces are constant or zero, then

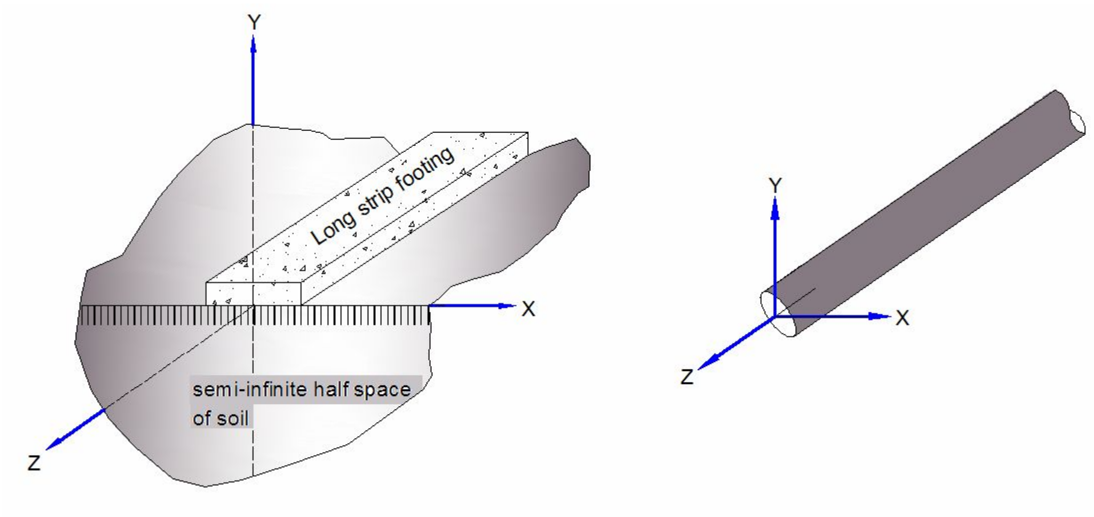
$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\sigma_x+\sigma_y\right)=0 \tag{5.2 a}$$

This equation of compatibility, combined with the equations of equilibrium, represents a useful form of the governing equations for problems of plane stress. The constitutive relation for such problems is given by

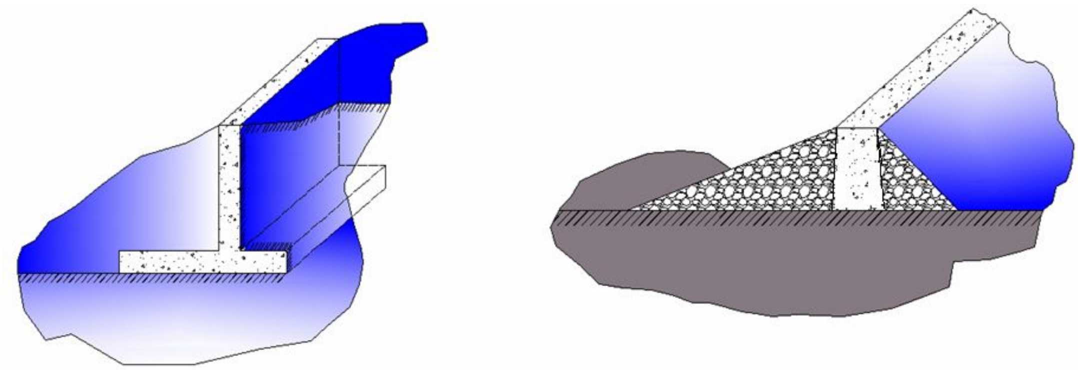
$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}=\frac{E}{\left(1-\nu^2\right)}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \left(\dfrac{1-\nu}{2}\right)\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} \tag{5.3}$$

Plane Strain Problems

Problems involving long bodies whose geometry and loading do not vary significantly in the longitudinal direction are referred to as plane-strain problems. Some examples of practical importance, shown in Figure 5.2, are a loaded semi-infinite half space such as a strip footing on a soil mass, a long cylinder; a tunnel; culvert; a laterally loaded retaining wall; and a long earth dam. In these problems, the dependent variables can be assumed to be functions of only the x and y co-ordinates, provided a cross-section is considered some distance away from the ends.



(a) Strip Footing (b) Long cylinder



(c) Retaining wall (d) Earth Dam

Figure 5.2 Examples of practical plane strain problems

Hence the strain components will be

$$\varepsilon_x = \frac{\partial u}{\partial x},\ \varepsilon_y = \frac{\partial v}{\partial y},\ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{5.4}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} = 0\,,\ \gamma_{xz} = \frac{\partial w}{\partial x}+\frac{\partial u}{\partial z} = 0\,,\ \gamma_{yz} = \frac{\partial w}{\partial y}+\frac{\partial v}{\partial z} = 0 \tag{5.5}$$

Moreover, from the vanishing of ε_z, the stress σ_z can be expressed in terms of σ_x and σ_y as

$$\sigma_z = \nu(\sigma_x + \sigma_y) \tag{5.6}$$

Compatibility Equation in terms of Stress Components (Plane strain case)

Stress-strain relations for plane strain problems are

$$\varepsilon_x = \frac{1}{E}\left[(1-\nu^2)\sigma_x - \nu(1+\nu)\sigma_y\right]$$

$$\varepsilon_y = \frac{1}{E}\left[(1-\nu^2)\sigma_y - \nu(1+\nu)\sigma_x\right] \tag{5.6 a}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

The equilibrium equations, strain-displacement elations and compatibility conditions are the same as for plane stress case also. Therefore substituting Eq. (5.6 a) in Eq. (5.1 a), we get

$$(1-\nu)\left[\frac{\partial^2\sigma_x}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial x^2}\right]-\nu\left[\frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial^2\sigma_x}{\partial x^2}\right] = 2\frac{\partial^2\tau_{xy}}{\partial x\partial y} \tag{5.6 b}$$

Now, differentiating the equilibrium equations (5.1 c) and (5.1 d) and adding the results as before and then substituting them in Eq. (5.6 b), we get

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_x+\sigma_y) = -\frac{1}{1-\nu}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right) \tag{5.6 c}$$

If the body forces are constant or zero, then

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_x+\sigma_y) = 0 \tag{5.6 d}$$

It can be noted that equations (5.6 d) and (5.2 a) are identical. Hence, if the body forces are zero or constant, the differential equations for plane strain will be same as that for plane stress. Further, it should be noted that neither the compatibility equations nor the equilibrium equations contain the elastic constants. Hence, the stress distribution is same for all isotropic materials in two dimensional state of stress. Also, the constitutive relation for plane strain problems is given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & \left(\frac{1-2\nu}{2}\right) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Relationship between plane stress and plane strain

(a) ***For plane-stress case***

From the stress-strain relationship (equation 4.20), we have

$\sigma_x = (2G+\lambda)\varepsilon_x + \lambda(\varepsilon_y + \varepsilon_z)$

or $\sigma_x = 2G\varepsilon_x + \lambda\varepsilon_x + \lambda(\varepsilon_y + \varepsilon_z)$

or $\sigma_x = \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G\varepsilon_x$

Similarly, $\sigma_y = \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G\varepsilon_y$

$$\sigma_z = \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G\varepsilon_z = 0$$

$$\tau_{xy} = G r_{xy} \qquad \tau_{yz} = G r_{yz} = 0 \qquad \tau_{zx} = G r_{zx} = 0$$

Denoting $(\varepsilon_x + \varepsilon_y + \varepsilon_z) = J_1 =$ First invariant of strain, then

$$\sigma_x = \lambda J_1 + 2G\varepsilon_x, \quad \sigma_y = \lambda J_1 + 2G\varepsilon_y, \quad \sigma_z = \lambda J_1 + 2G\varepsilon_z = 0 \qquad \text{(a)}$$

From, $\sigma_z = 0$, we get

$$\varepsilon_z = -\frac{\lambda}{(\lambda + 2G)}(\varepsilon_x + \varepsilon_y)$$

Using the above value of ε_z, the strain invariant J_1 becomes

$$J_1 = \frac{2G}{\lambda + 2G}(\varepsilon_x + \varepsilon_y) \qquad \text{(b)}$$

Substituting the value of J_1 in equation (a), we get

$$\sigma_x = \frac{2G\lambda}{\lambda + 2G}\left(\varepsilon_x + \varepsilon_y\right) + 2G\varepsilon_x$$

$$\sigma_y = \frac{2G\lambda}{\lambda + 2G}\left(\varepsilon_x + \varepsilon_y\right) + 2G\varepsilon_y$$

(b) ***For plane-strain case***

Here $\varepsilon_z = 0$

$$\therefore \sigma_x = \lambda J_1 + 2G\varepsilon_x = \lambda\left(\varepsilon_x + \varepsilon_y\right) + 2G\varepsilon_x$$

$$\sigma_y = \lambda J_1 + 2G\varepsilon_y = \lambda\left(\varepsilon_x + \varepsilon_y\right) + 2G\varepsilon_y$$

$$\sigma_z = \lambda J_1 \qquad = \lambda\left(\varepsilon_x + \varepsilon_y\right)$$

If the equations for stress σ_x for plane strain and plane stress are compared, it can be observed that they are identical except for the comparison of co-efficients of the term $\left(\varepsilon_x + \varepsilon_y\right)$.

i.e., $$\sigma_x = \begin{cases} \lambda\left(\varepsilon_x + \varepsilon_y\right) + 2G\varepsilon_x & \text{plane strain} \\ \dfrac{2G\lambda}{\lambda + 2G}\left(\varepsilon_x + \varepsilon_y\right) + 2G\varepsilon_x & \text{plane stress} \end{cases}$$

Since all the equations for stresses in plane-stress and plane-strain solutions are identical, the results from plane strain can be transformed into plane stress by replacing λ in plane-strain case by $\dfrac{2G\lambda}{\lambda + 2G}$ in plane-stress case. This is equivalent to replacing $\dfrac{\nu}{1-\nu}$ in plane strain case by ν in plane stress case. Similarly, a plane-stress solution can be transformed into a plane-strain solution by replacing $\dfrac{2G\lambda}{\lambda + 2G}$ in plane-stress case by λ in plane-strain case. This is equivalent to replacing ν in plane-stress case by $\dfrac{\nu}{1-\nu}$ in plane-strain case.