MECHANICAL ENGINEERING DEPARTMENT

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SUBJECT: MECHANICS OF MATERIALS II **SEMESTER: 4^{th} (Spring)**

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CLASS-NOTES

Energy Methods

Strain Energy

Strain Energy of the member is defined as the internal work done in defoming the body by the action of externally applied forces. This energy in elastic bodies is known as **elastic strain energy** :

Strain Energy in uniaxial Loading

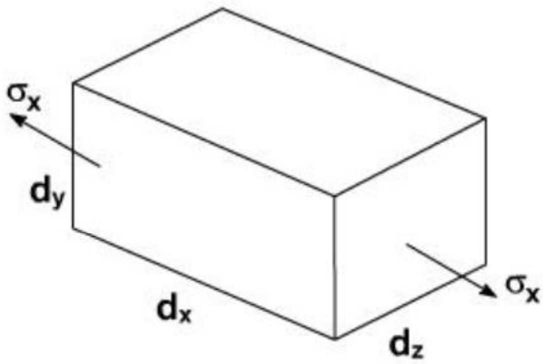


Fig .1

Let as consider an infinitesimal element of dimensions as shown in Fig .1. Let the element be subjected to normal stress σ_x.

The forces acting on the face of this element is σ_x. dy. dz

where

dydz = Area of the element due to the application of forces, the element deforms to an amount = $\in_x$ dx

□ $\in_x$ = strain in the material in x – direction

$$= \frac{\text{Change in length}}{\text{Orginal in length}}$$

Assuming the element material to be as linearly elastic the stress is directly proportional to strain as shown in Fig . 2.

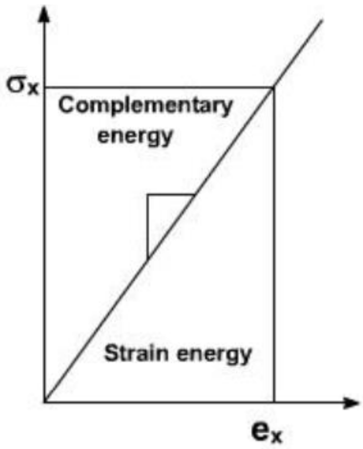


Fig .2

∴ From Fig .2 the force that acts on the element increases linearly from zero until it attains its full value.

Hence average force on the element is equal to ½ σ_x . dy. dz.

∴ Therefore the workdone by the above force

Force = average force x deformed length

= ½ σ_x. dydz . $\in_x$. dx

For a perfectly elastic body the above work done is the internal strain energy "du".

$$\boxed{du = \frac{1}{2}\sigma_x dydz \in_x dx} \qquad .....(2)$$

$$= \frac{1}{2}\sigma_x \in_x dxdydz$$

$$\boxed{du = \frac{1}{2}\sigma_x \in_x dv} \qquad ....(3)$$

where dv = dxdydz

= Volume of the element

By rearranging the above equation we can write

$$U_o = \boxed{\frac{du}{dv} = \frac{1}{2}\sigma_x \in_x} \qquad .....(4)$$

The equation (4) represents the strain energy in elastic body per unit volume of the material its strain energy – density 'u_o' .

From Hook's Law for elastic bodies, it may be recalled that

$$\boxed{\sigma = E\epsilon}$$

$$\boxed{U_o = \frac{du}{dv} = \frac{\sigma_x^{\,2}}{2E} = \frac{E\,\epsilon_x^{\,2}}{2}} \qquad \text{......(5)}$$

$$\boxed{U = \int_{Vol} \frac{\sigma_x^{\,2}}{2E}\,dv} \qquad \text{.......(6)}$$

In the case of a rod of uniform cross – section subjected at its ends an equal and opposite forces of magnitude P as shown in the Fig .3.

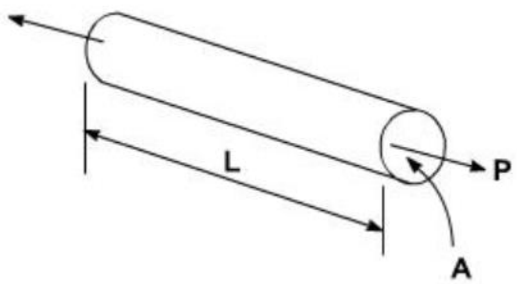


Fig .3

$$U = \int_{Vol}^{L} \frac{\sigma_x^{\,2}}{2E}\,dv \qquad \sigma_x = \frac{P}{A}$$

$$U = \int_0^L \frac{P^2}{2EA^2}\,A\,dx \qquad dv = Adx = \text{Element volume}$$

A = Area of the bar.

L = Length of the bar

$$\boxed{U = \frac{P^2L}{2AE}} \qquad \text{.....(7)}$$

Modulus of resilience :

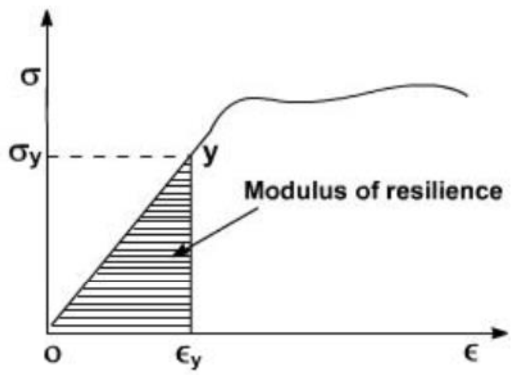


Fig .4

Suppose ' σ_x' in strain energy equation is put equal to σ_y i.e. the stress at proportional limit or yield point. The resulting strain energy gives an index of the materials ability to store or absorb energy without permanent deformation

So $$\boxed{U_y = \frac{\sigma_y^2}{2E}} \qquad .....(8)$$

The quantity resulting from the above equation is called the Modulus of resilience

The modulus of resilience is equal to the area under the straight line portion 'OY' of the stress – strain diagram as shown in Fig .4 and represents the energy per unit volume that the material can absorb without yielding. Hence this is used to differentiate materials for applications where energy must be absorbed by members.

Modulus of Toughness :

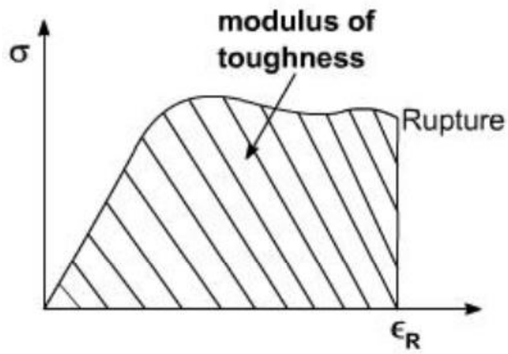


Fig .5

Suppose 'ϵ' [strain] in strain energy expression is replaced by ϵ_R strain at rupture, the resulting strain energy density is called modulus of toughness

$$U = \int_0^{\epsilon} E\,\epsilon_x \; dx = \frac{E\,\epsilon_R^2}{2} dv$$

$$\boxed{U = \frac{E\,\epsilon_R^2}{2}} \qquad ......(9)$$

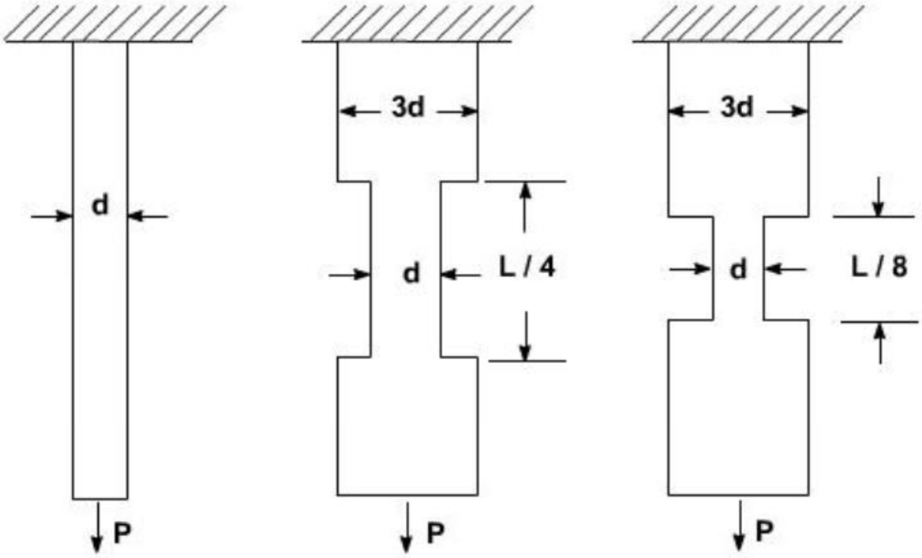
From the stress – strain diagram, the area under the complete curve gives the measure of modules of toughness. It is the materials.

Ability to absorb energy upto fracture. It is clear that the toughness of a material is related to its ductility as well as to its ultimate strength and that the capacity of a structure to withstand an impact Load depends upon the toughness of the material used.

ILLUSTRATIVE PROBLEMS

1. Three round bars having the same length 'L' but different shapes are shown in fig below. The first bar has a diameter 'd' over its entire length, the second had this diameter over one – fourth of its length, and the third

has this diameter over one eighth of its length. All three bars are subjected to the same load P. Compare the amounts of strain energy stored in the bars, assuming the linear elastic behavior.



Solution :

1.The strain Energy of the first bar is expressed as

$$U_1 = \frac{P^2L}{2EA}$$

2.The strain Energy of the second bar is expressed as

$$U_2 = \frac{P^2(L/4)}{2EA} + \frac{P^2(3L/4)}{2E9A} = \frac{P^2L}{6EA}$$

$$\boxed{U_2 = \frac{U_1}{3}}$$

3.The strain Energy of the third bar is expressed as

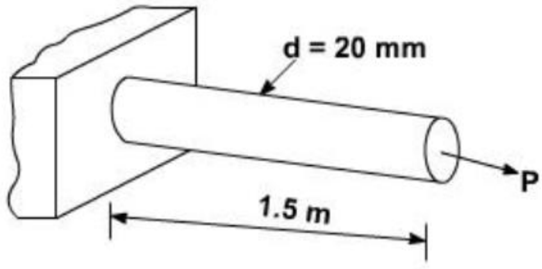
$$U_3 = \frac{P^2(L/8)}{2EA} + \frac{P^2(7L/8)}{2E(9A)}$$

$$U_3 = \frac{P^2L}{9EA}$$

$$\boxed{U_3 = \frac{2U_1}{9}}$$

From the above results it may be observed that the strain energy decreases as the volume of the bar increases.

2. Suppose a rod AB must acquire an elastic strain energy of 13.6 N.m using E = 200 GPa. Determine the required yield strength of steel. If the factor of safety w.r.t. permanent deformation is equal to 5.



Solution :

Factor of safety = 5

Therefore, the strain energy of the rod should be u = 5 [13.6] = 68 N.m

Strain Energy density

The volume of the rod is

$$V = AL = \frac{\pi}{4} d^2 L$$
$$= \frac{\pi}{4} 20 \times 1.5 \times 10^3$$
$$= 471 \times 10^3 \ mm^3$$

Yield Strength :

As we know that the modulus of resilience is equal to the strain energy density when maximum stress is equal to σ_x .

$$U = \frac{\sigma_y^2}{2E}$$
$$0.144 = \frac{\sigma_y^2}{2 \times (200 \times 10^3)}$$
$$\boxed{\sigma_y = 200 \, \text{Mpa}}$$

It is important to note that, since energy loads are not linearly related to the stress they produce, factor of safety associated with energy loads should be applied to the energy loads and not to the stresses.

Strain Energy in Bending :

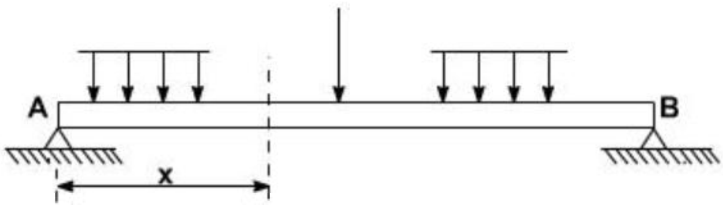


Fig .6

Consider a beam AB subjected to a given loading as shown in figure.

Let

M = The value of bending Moment at a distance x from end A.

From the simple bending theory, the normal stress due to bending alone is expressed as.

$$\boxed{\sigma = \frac{MY}{I}}$$

Substituting the above relation in the expression of strain energy

i.e. $U = \int \frac{\sigma^2}{2E} dv$

$$= \int \frac{M^2 . y^2}{2EI^2} dv \qquad .....(10)$$

Substituting dv = dxdA

Where dA = elemental cross-sectional area

$\frac{M^2 . y^2}{2EI^2} \rightarrow$ is a function of x alone

Now substitiuting for dy in the expression of U.

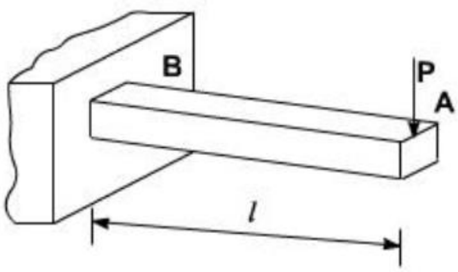
$$U = \int_0^L \frac{M^2}{2EI^2} \left(\int y^2 dA \right) dx \qquad .....(11)$$

We know $\int y^2 dA$ represents the moment of inertia 'I' of the cross-section about its neutral axis.

$$\boxed{U = \int_0^L \frac{M^2}{2EI} dx} \qquad .....(12)$$

ILLUSTRATIVE PROBLEMS

1. Determine the strain energy of a prismatic cantilever beam as shown in the figure by taking into account only the effect of the normal stresses.



Solution : The bending moment at a distance x from end
A is defined as

$$\boxed{M = -Px}$$

Substituting the above value of M in the expression of strain energy we may write

$$U = \int_0^L \frac{P^2 x^2}{2EI} dx$$

$$\boxed{U = \int_0^L \frac{P^2 L^3}{EI}}$$

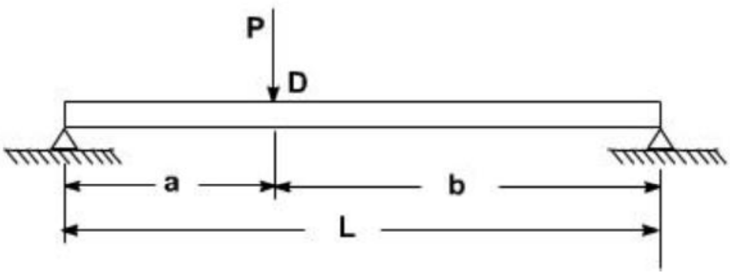
Problem 2 :

a. Determine the expression for strain energy of the prismatic beam AB for the loading as shown in figure below. Take into account only the effect of normal stresses due to bending.
b. Evaluate the strain energy for the following values of the beam

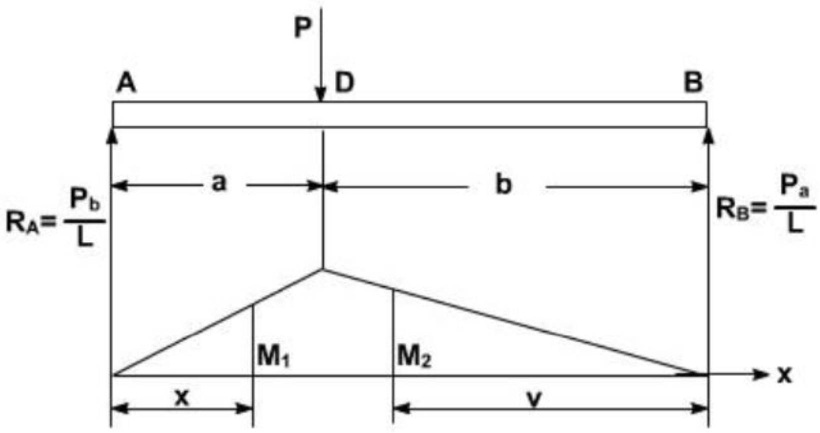
P = 208 KN ; L = 3.6 m = 3600 mm

A = 0.9 m = 90mm ; b = 2.7m = 2700 mm

E = 200 GPa ; I = 104 x 10^8 mm^4



Solution:



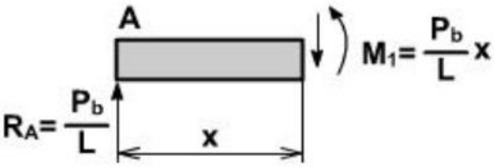
a.

Bending Moment : Using the free – body diagram of the entire beam, we may determine the values of reactions as follows:

$R_A = P_b / L$ $R_B = P_a / L$

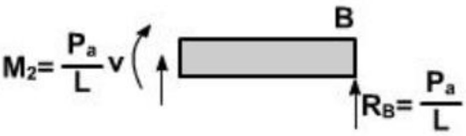
For Portion AD of the beam, the bending moment is

$$\boxed{M_1 = \frac{P_b}{L} x}$$



For Portion DB, the bending moment at a distance v from end B is

$$\boxed{M_2 = \frac{P_a}{L} x}$$



Strain Energy :

Since strain energy is a scalar quantity, we may add the strain energy of portion AD to that of DB to obtain the total strain energy of the beam.

$$U = U_{AD} + U_{DB}$$

$$= \int_0^a \frac{M_1^2}{2EI} dx + \int_0^b \frac{M_2^2}{2EI} dv$$

$$= \frac{1}{2EI} \int_0^a \left(\frac{P_b}{L} x \right)^2 dx + \frac{1}{2EI} \int_0^b \left(\frac{P_a}{L} v \right)^2 dx$$

$$= \frac{1}{2EI} \frac{P^2}{L^2} \left(\frac{b^2 a^3}{3} + \frac{a^2 b^3}{3} \right)$$

$$\boxed{U = \frac{P^2 a^2 b^2}{6EIL^2} (a + b)}$$

$$\text{Since} (a + b) = L$$

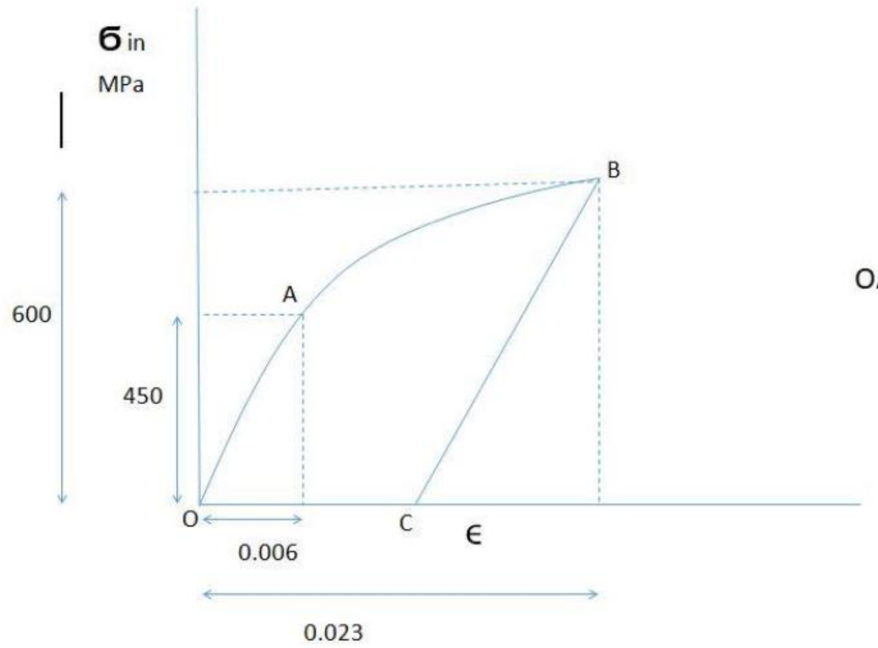
$$\boxed{U = \frac{P^2 a^2 b^2}{6EIL}}$$

b. Substituting the values of P, a, b, E, I, and L in the expression above.

$$U = \frac{(200 \times 10^3)^2 \times (900)^2 \times (2700)^2}{6 (200 \times 10^3) \times (104 \times 10^6) \times (3600)} = 5.27 \times 10^7 \text{ KN.m}$$

Ex: Stress strain diagram for aluminium alloy is shown in figure. If the specimen of the material is stressed to 600 MPa. Determine the permanent strain that remains in specimen when the load is released. Also calculate the modulus of resilience both before and after the load application?

Sol:



OA & BC are parallel

Slope = E = 450/0.06 = 75000MPa

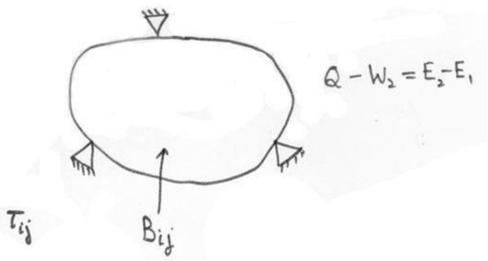
Also, E = 600 / (0.023 - ϵ_{OC}) = 75000

ϵ_{OC} = **0.015**

$u_{initial} = \frac{1}{2} * 450 * 0.006$ = 1.35 **MJ/m³**

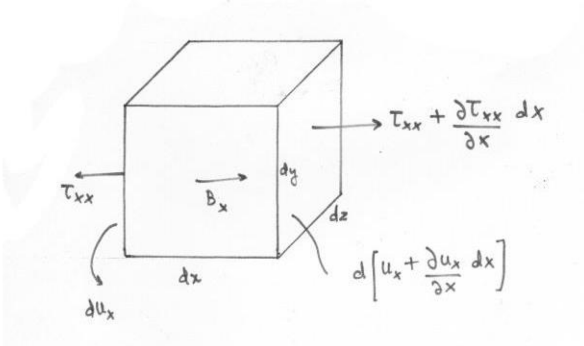
$u_{final} = \frac{1}{2} * 600 * 8 * 10^{-3}$ = 2.40 **MJ/m³**

- **Q-W=E_1-E_2**



#E=U + B.P.E + B.K.E

- **Strain Energy – stress strain distribution.**



The increment of work done by the stresses τ_{xx} on face 1 and by $\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x} dx$ on face 2 during infinitesimal displacement du_x at face 1 and $d\left[u_x + \frac{\partial u_x}{\partial x} dx\right]$ at face 2 is:

$$W = -\tau_{xx}\, du_x\, dy\, dz + \left(\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x}\, dx\right) d\left[u_x + \frac{\partial u_x}{\partial x}\, dx\right] dy\, dz$$
$$+ B_x dx\, dy\, dz\;\; d\left[u_x + \alpha\, \frac{\partial u_x}{\partial x}\, dx\right]$$

Using body force component $\mathbf{B_x}$ and $\boldsymbol{\alpha}$ is some fraction

On solving and neglecting higher order terms we get,

$$\text{W=}\left\{\tau_{xx}\, d\left(\frac{\partial u_x}{\partial x}\right) + \partial\left(\frac{\partial \tau_{xx}}{\partial x} + B_x\right) du_x\right\} d_x d_y d_z$$

Replace $\frac{\partial u_x}{\partial x} = \in_{xx}$ and also equilibrium requirement: $\frac{\partial \tau_{xx}}{\partial x} + B_x = 0$

$$\boxed{\mathbf{W=}\boldsymbol{\tau_{xx}}\mathbf{d}\boldsymbol{\in_{xx}}\mathbf{.dV}}$$

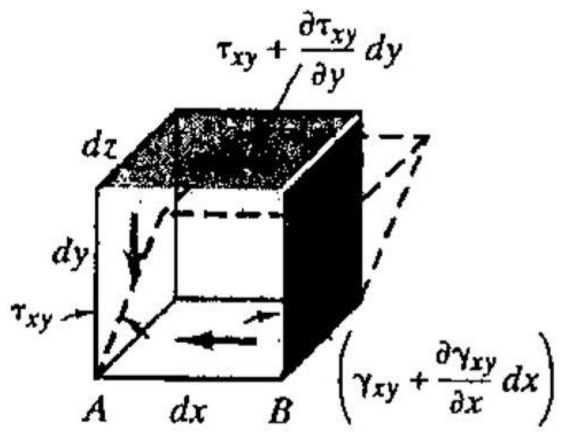
Considering normal stresses and strains in the **Y** and **Z** directions; similar expressions for work increments done on the elements can be obtained.

For a state of stress on the element wherein $\tau_{xx},\ \tau_{yy}$ and τ_{zz} act simultaneously, we may compute the work increment on the element by superposing the independent work increments of the three directions.

For an elastic material, this represents the strain energy increment of the element resulting from normal strains.

$$(\boldsymbol{\tau_{xx}}\mathbf{d}\boldsymbol{\in_{xx}} + \boldsymbol{\tau_{yy}}\mathbf{d}\boldsymbol{\in_{yy}} + \boldsymbol{\tau_{zz}}\mathbf{d}\boldsymbol{\in_{zz}})\mathbf{dV}$$

As next step, let us consider strain energy associated with **shear strain**



Work increment:

$$\left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} dy d\gamma_{xy}\right) dxdzd\left(\gamma_{xy} + \frac{\partial \gamma_{xy}}{\partial x} \beta . dx\right) dy$$

β = 0 at LS and β = 1 at RS

$$\tau_{xy} d\gamma_{xy} + \tau_{xy} d\left(\frac{\partial \gamma_{xy}}{\partial x}\right) \beta . dx + \frac{\partial \tau_{xy}}{\partial y} dy d\gamma_{xy} + \frac{\partial \tau_{xy}}{\partial y} d\left(\frac{\gamma_{xy}}{\partial x}\right) \beta dxdy$$

The expression for increment of work done by shear stress on element is then

$$\boxed{(\tau_{xy} d\gamma_{xy}) dxdydz}$$

Similarly, considering the increments of work done by shear stress τ_{xz} & τ_{yz} and noting we can superpose the results for an isotropic elastic materials, the total strain energy increment resulting from shear deformation for the element is then

$$[\tau_{xy} d\gamma_{xy} + \tau_{xz} d\gamma_{xz} + \tau_{yz} d\gamma_{yz}] dV$$

Since normal stresses do no work as a result of shear strains and shear stresses do no work as a result of normal strain.

For elastic isotropic material, the total strain energy increment for an element under general state of stress

$$dU = (\tau_{xx} d\epsilon_{xx} + \tau_{yy} d\epsilon_{yy} + \tau_{zz} d\epsilon_{zz} + \tau_{xy} d\gamma_{xy} + \tau_{xz} d\gamma_{xz} + \tau_{yz} d\gamma_{yz}) dV$$

$$\boxed{d\text{Ꞷ} = \sum_i \sum_j \tau_{ij} d\epsilon_{ij}}$$

For linear, elastic and Isotropic material:

$d\text{Ա} = \frac{\tau_{xx}}{E}[d\tau_{xx} - \mu(d\tau_{yy} + d\tau_{zz})] + \frac{\tau_{yy}}{E}[d\tau_{yy} - \mu(d\tau_{xx} + d\tau_{zz})] + \frac{\tau_{zz}}{E}[d\tau_{zz} - \mu(d\tau_{xx} + d\tau_{yy})] + \frac{\tau_{xy}}{G}d\tau_{xy} + \frac{\tau_{xz}}{G}d\tau_{xz} + \frac{\tau_{yz}}{G}d\tau_{yz}$

$d\text{Ա} = \frac{1}{E}[\tau_{xx}dx + \tau_{yy}dy + \tau_{zz}dz] - \frac{\mu}{E}[\tau_{xx}d\tau_{yy} + \tau_{xx}d\tau_{zz}] + \frac{1}{G}[\tau_{xy}d\tau_{xy} + \frac{\tau_{xz}}{G}d\tau_{xz} + \tau_{yz}d\tau_{yz}]$

$$\boxed{\#\ \text{Ա} = \frac{1}{2E}[\tau^2_{xx} + \tau^2_{yy} + \tau^2_{zz}] - \frac{\mu}{E}[\tau_{xx}\tau_{yy} + \tau_{zz}\tau_{xx} + \tau_{yy}\tau_{zz}] + \frac{1}{2G}[\tau^2_{xy} + \tau^2_{yz} + \tau^2_{xz}]}$$

$$\boxed{\#\ \text{Ա} = \frac{E\mu}{2(1+\mu)(1-2\mu)}(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz})^2 + G(\epsilon^2_{xx} + \epsilon^2_{yy} + \epsilon^2_{zz}) + \frac{G}{2}(\gamma^2_{xy} + \gamma^2_{yz} + \gamma^2_{xz})}$$

Ex: What is the strain energy per unit volume of a point for a linear, elastic material having at this point the following state of stress?

$$\tau_{ij} = \begin{pmatrix} 1000 & -500 & 2000 \\ -500 & 2000 & -400 \\ 2000 & -400 & 1000 \end{pmatrix} \text{Psi} \qquad G = 15\times10^6 \text{ psi},\ E = 3\times10^6 \text{ psi}$$

Sol: $E = 2G(1+\mu)$

$30\times10^6 = (15\times10^6)(2)[1+\mu]$

$\mu + 1 = 1$

$\mu = 0$

$\text{Ա} = \frac{1}{2}(3\times10^6)[(1000)^2 + (2000)^2 + (1000)^2] + \frac{1}{30}\times10^6[(500)^2 + (2000)^2 + (400)^2]$

$\text{Ա} = 1/60\times10^6[60\times10^6 + (882\times10^4)]$

$\text{Ա} = 1/60\times10^6[6 + 8.82]$

$\text{Ա} = 0.247$ lb-in/in^3

Ex: What is the S.E per unit volume at a point for linear, elastic material having following:

$$\epsilon_{ij} = \begin{pmatrix} 0.001 & -0.0005 & 0.003 \\ -0.0005 & 0.002 & -0.002 \\ 0.003 & -0.002 & -0.001 \end{pmatrix}$$

$G=1\text{x}10^{11}$ psi & $\mu = 0.3$

Sol:

$E=2\text{x}10^{11}[1.3]=2.6\text{x}10^{11}$ Pa

$$\upsilon = \frac{1}{2}\frac{\left(2.6\times 10^{11}\right)(0.3)(0.002)^2}{(1.3)(0.4)} + 10^{11}[(0.001)^2 + (0.002)^2 + (0.001)^2]$$

$$+ \frac{1}{2}\times 10^{11}[(0.005)^2 + (0.002)^2 + (0.003)^2]$$

$$\upsilon = 3\times 10^5 + 6\times 10^5 + 2.65\times 10^6$$

$\boldsymbol{\upsilon = 3.55\times 10^6 \text{Nm/m}^3}$ → Energy per unit volume

Alternate method:

$$\tau_{ij} = \lambda J_1 \delta_{ij} + 2G\epsilon_{ij}$$

$$\lambda = \frac{E\mu}{(1+\mu)(1-2\mu)}$$

$J_1 = 0.002$, $\lambda = 1\text{x}10^{11}$ Pa

1) $\tau_{xx} = (1.5\times 10^{11})(0.002) + (2\times 10^{11})(0.001)$

=> $\tau_{xx} = 5\times 10^8$Pa

2) $\tau_{yy} = (1.5\times 10^{11})(0.002) + (2\times 10^{11})(0.001)$

=> $\tau_{yy} = 7\times 10^8$Pa

3) $\tau_{zz} = 1\times 10^8$

4) $\tau_{xy} = 2\times 10^{11}(-0.005) = -10^8$Pa

5) $\tau_{xz} = 2\times 10^{11}(-0.001) = 6\times 10^8$Pa

6) $\tau_{yz} = 2\times 10^{11}(0.002) = -4\times 10^8$Pa

$$\tau_{ij} = \begin{bmatrix} 5 & -1 & 6 \\ -1 & 7 & -4 \\ 6 & -4 & 1 \end{bmatrix}\times 10^8 \text{ Pa}$$

$$u = \frac{1}{2\times 2.6\times 10^{11}}[25+49+1]\times 10^{16} - \frac{0.3}{2.6\times 10^{11}}\times(35+5+7)\times 10^{11} + \frac{1}{2\times 10^{11}}[1+36+16]\times 10^{16}$$

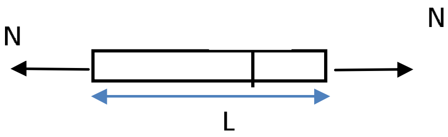
$$u = 14.42\times 10^{5} - 5.4\times 10^{5} + 26.5\times 10^{5}\ \ \mathrm{Nm/m^3}$$

$$\mathbf{u = 3.55\times 10^{6}\,Nm/m^3}$$

- **STRAIN ENERGY FOR VARIOUS TYPES OF LOADING:**

1. AXIAL LOAD:

$$U = \int_V \frac{\sigma_x^2}{2E}\, dV$$

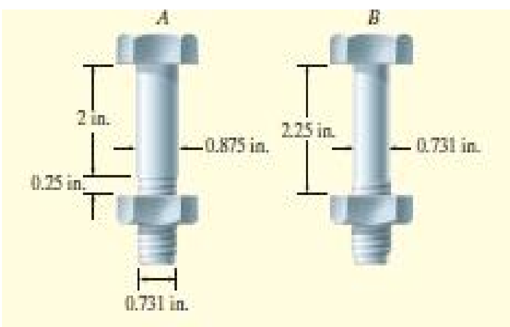


$$U = \int_V \frac{N^2}{2EA^2}\ Adx$$

$$\boxed{U = \frac{N^2L}{2AE}}$$

Ex:
One of the two high-strength steel bolts *A* and *B* shown in Fig. is to be chosen to support a sudden tensile loading. For the choice it is Necessary to determine the greatest amount of elastic strain energy That each bolt can absorb. Bolt *A* has a diameter of 0.875 in. for 2 in. Of its length and a root (or smallest) diameter of 0.731 in. within the 0.25-in. threaded region. Bolt *B* has "upset" threads, such that the Diameter throughout its 2.25-in. length can be taken as 0.731 in. In Both cases, neglect the extra material that makes up the threads. Take

$E_{st} = 29(10^3)$ ksi & ρ_y=44ksi



Sol:

Bolt A: If the bolt is subjected to its maximum tension, the Maximum stress of ρ_y= 44 ksi will occur within the 0.25-in. region. This tension force is

$$P_{max}=\rho_y A=44ksi\left[\pi\ \left(\frac{0.731in.}{2}\right)^2\right]=18.47kip$$

$$U_A=\sum\frac{N^2L}{2AE}=\frac{N^2}{2E}\left[\frac{L_1}{A_1}+\frac{L_2}{A_2}\right]$$

$$U_A=\frac{18.45^2}{2(29)(10^3)}\left[\frac{3.08}{\frac{\pi}{4}}\right]=\mathbf{0.023\ klb\text{-}in}$$

Bolt B :

$$U_B= N^2L/2EA$$

$$U_B = (18.46)^2(2.25)/2*\left(\frac{\pi}{4}\right)(.731)^2 * (29*10^3)$$

$$U_B = 0.0315\ \text{in-klb}$$

As, $U_B > U_A$, i.e., Bolt 'B' can absorb more energy. Hence Bolt 'B' should be chosen.

% Strength = $(U_B-U_A)/U_A$ = 36 %

2. BENDING MOMENT:

$$U = \iiint\left(\frac{\sigma^2}{2E}\right)dv$$

$$\boldsymbol{\sigma} = \frac{My}{I}$$

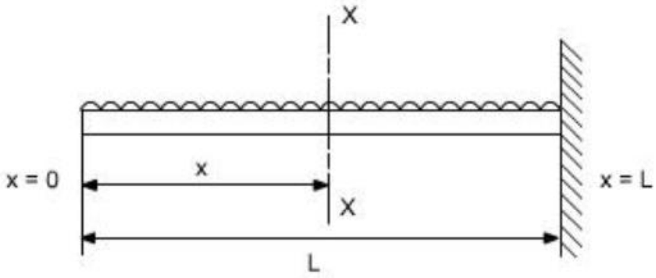
Where, M = Bending Moment

& I = Moment of Inertia or 2nd M.O.A

$$U = \int_0^L(M^2/2EI^2)dx\,.\iint y^2dA$$

$$\boxed{\mathbf{U} = \int_0^L\frac{\mathbf{M^2}}{\mathbf{2EI}}dx}$$

Ex: Determine the Elastic strain energy due to bending of the cantilever beam, if the beam is subjected to a uniformly distributed load (UDL) 'W' & (E.I) is constant.

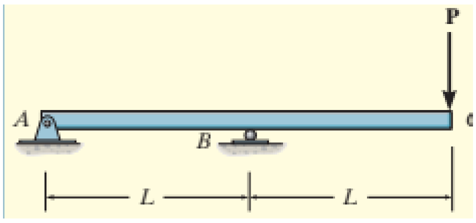


Sol: Here, $M = \frac{-Wx^2}{2}$

$$\therefore U = \int_0^L \frac{W^2x^4}{2E(4)I} dx = \frac{W^2}{8EI}\int_0^L x^4 dx = \frac{W^2}{40\ EI}[x^5]_0^L$$

$$\therefore \mathbf{U} = \frac{\mathbf{W^2L^5}}{\mathbf{40\ EI}}$$

Ex: Determine the Bending strain energy in the region AB of the beam



M = -px

$$U = \int_0^L \frac{(-px)^2}{2EI} dx$$

$$U = \frac{p^2x^3}{6EI}$$

$$\mathbf{U} = \frac{\mathbf{p^2l^3}}{\mathbf{6EI}}$$

3._TRANSVERSE SHEAR:

$$U_i = \iiint \frac{\tau^2}{2G} dv$$

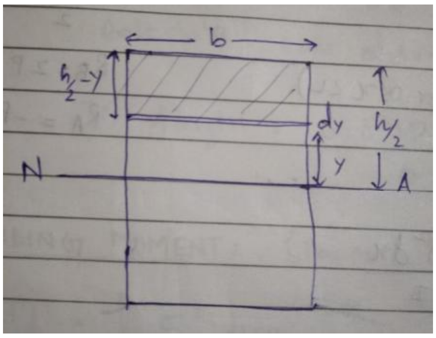
$$\tau = \frac{V_y Q_z}{I_{zz} b}$$ (Beam has constant cross. area)

$$Ui = \int_0^L \frac{V_y 2}{2GI_{zz}\text{^}2} \left(\int \frac{Q_z 2}{b^2} dA\right) dx$$

$$f_s = \frac{A}{Izz^2} \int \frac{Q_z 2}{b^2} dA$$

$$\boxed{Ui = \int_0^L \frac{f_s V_y 2}{2GA} dx}$$

Ex:



A=bh

Q_{zz}=(y + [h/2 -y] /2) b (h/2 - y)

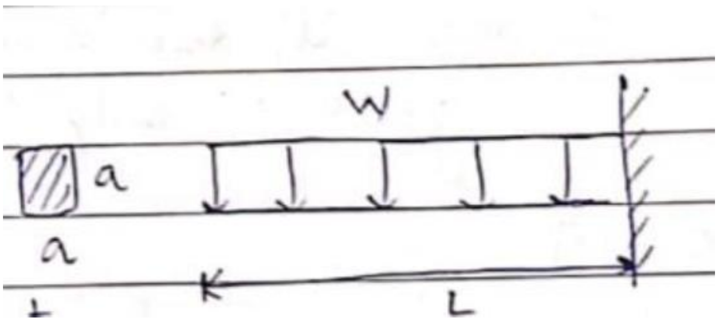
Q_{zz} = [b/2] [h2/4 -y^2]

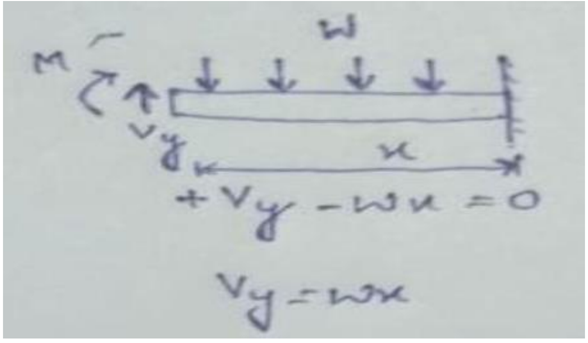
$$f_S = \frac{bh}{\left(\frac{bh^3}{12}\right)^2} \int_{-h/2}^{h/2} \frac{b^2\left(\frac{h^2}{4} - Y^2\right)^2 b\,dy}{4b^2}$$

On integrating we get

f_s= 6/5

Ex: Determine the strain energy, where EI is constant and f_s = 6/5





$V_y = Wx$

$U_i = {}_0^L\int (fs.Vy^2/2Ga^2).dx$

$=6/10Ga^2 \; {}_0^L\int (Wx)^{2.dx}$

$$\mathbf{U_i = (\frac{1}{5Ga\text{^}2}).W^2L^3}$$

Now,

$\frac{(U_i)_s}{(U_i)_b} = (W^2L^3/5GA) / (W^2L^5/40EI)$

$= (8EI/GA).1/L^2$ (as $E= 2G (1+\mu)$)

$= 2/3(a/L)^2. (E/G)$

Since we know that higher value of E=3G

$\frac{(U_i)_s}{(U_i)_b} = 2. (a/L)^2$

If L=5a (min. length of beam)

$\frac{(U_i)_s}{(U_i)_b} = \frac{2}{5}$

$$\boxed{\frac{(U_i)_s}{(U_i)_b} = 8\%}$$

=> Maximum strain energy due to shear stress is only 8% of the strain energy due to bending moment.

∴ we neglect strain energy due to shear stress.

4. TORSIONAL MOMENT

$$U_i = \iiint \frac{\tau^2}{2G}.dv$$

From torsion equation: $\tau = \frac{Tr}{J}$

$$\therefore U_i = \iiint \frac{(T.r)^2}{2GJ^2}.dAdx$$

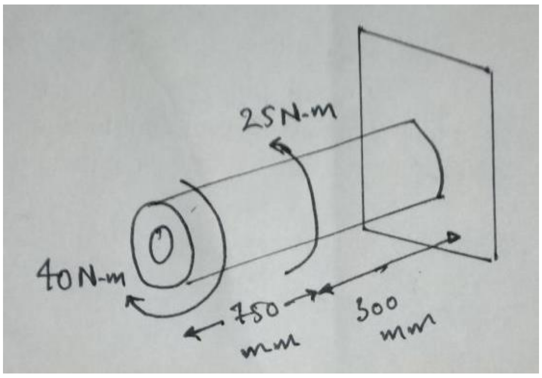
$$= \int \frac{T^2}{2GJ^2}.dx \int \frac{r^2}{1} dA$$

$$= \int \frac{T^2}{2GJ}.dx \quad \text{(over length L)}$$

$$\boxed{U_i = \frac{T^2 L}{2GJ}}$$

Ex: A tabular shaft in figure is fixed to the wall and subjected to two torques as shown. Calculate the strain energy stored in shaft due to this loading.

r_0=80mm, G=75Gpa, r_i=65mm.



$U = \Sigma T^2 L/2GJ$

$J = \frac{\pi}{2}[(80)^2 - (65)^2]$

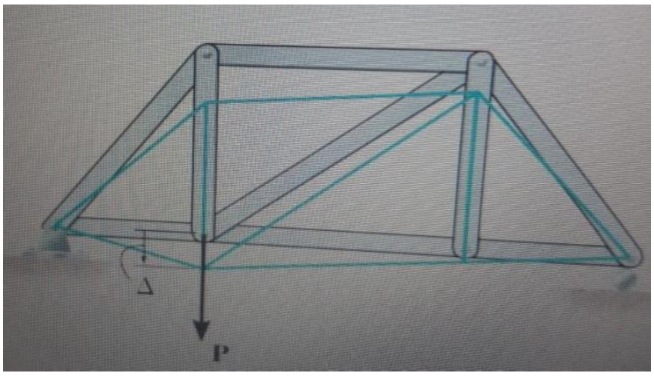
$$U = \frac{(40)^\wedge 2\ (0.75)}{2*75*10^3*36.3} + \frac{(15)^\wedge 2(0.3)}{2*75*10^3*36.3}$$

U=233μJ

NOTE: If a loading is applied slowly to a body, the external load tends to deform the body. So, the loads tend to do external work Ue, as they are displaced. This external works caused by the loads is transformed into internal works or strain Energy Ui, which is stored in the body.

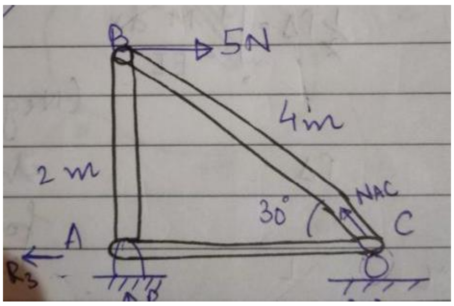
- **CONSERVATION OF ENERGY :**

- Ue = Ui (up to elastic limit)



$$\frac{1}{2}P\Delta = \sum \frac{N^2L}{2AE}$$ [($N^2L/2AE$) = strain energy by stress distribution

Ex: The three bar truss is subjected to a horizontal force 5N, if the cross sectional area of each member is 0.2m^2. Determine the horizontal displacement of point B. E= 29×10^3Pa



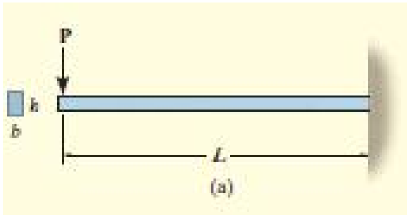
Member	Length(L)	Force(N)	N^2L
AB	2	2.88	16.59
BC	4	5.77	133.17
CA	$2\sqrt{3}$	5	86.6

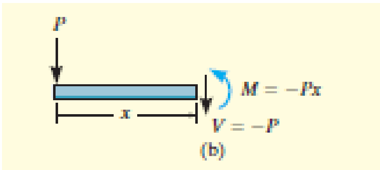
$$\sum N^2L = 236.36$$

Now, $\frac{1}{2}P\Delta = \sum \frac{N^2L}{2AE}$ (P=5N)

Δ = 8.14mm

Ex: The cantilevered beam in Fig. has a rectangular cross section and is subjected to a load **P** at its end. Determine the displacement of the load. *EI* is constant.





$$\frac{1}{2}P\Delta = \int_0^L \frac{f_s\, V^2\, dx}{2GA} + \int_0^L \frac{M^2\, dx}{2EI}$$

$$= \frac{3P^2L}{5GA} + \frac{P^2L^3}{6EI}$$

Now, $\frac{3}{5}\frac{P^2L}{GA} \leq \frac{p^2L^3}{6EI}$

$$\frac{3}{5}\frac{P^2L}{G(bh)} \leq \frac{p^2L^3}{6E\left[\frac{1}{12}(bh^3)\right]}$$

$$\frac{3}{5G} \leq \frac{2L^2}{Eh^2}$$

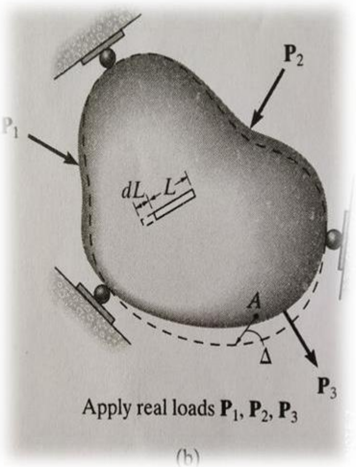
Since E ≤ 3G, then $0.9 \leq \left(\frac{L}{h}\right)^2$ hence neglect this part.

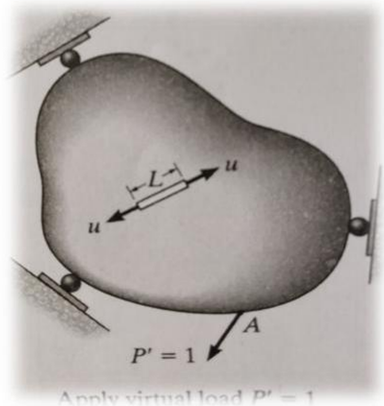
$$\frac{1}{2}P\Delta = \frac{p^2L^3}{6EI}$$

$\Delta = \boldsymbol{\frac{pL^3}{3EI}}$ **ans.**

- **PRINCIPLE OF VIRTUAL WORK**

$$\sum P\Delta = \sum U\delta$$

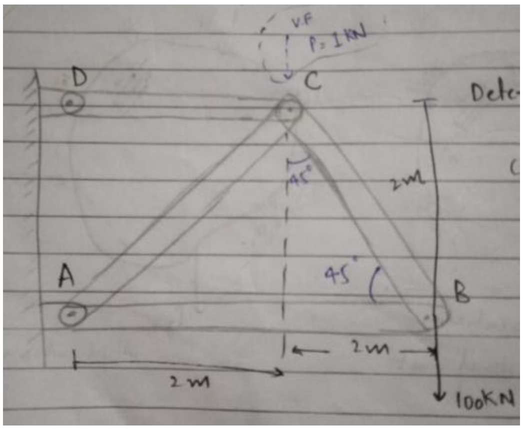




$$\boxed{\Delta = \sum \boldsymbol{u.dL}}$$

DEFORMATION CAUSED BY	STRAIN ENERGY	INTERNAL VIRTUAL WORK
Axial Load **N**	$\int_0^L \frac{N^2}{2EA}dx$	$\int_0^L \frac{Nn}{EA}dx$
Shear force **V**	$\int_0^L \frac{fsV^2}{2GA}dx$	$\int_0^L \frac{fvV}{GA}dx$
Bending Moment **M**	$\int_0^L \frac{M^2}{2EI}dx$	$\int_0^L \frac{mM}{EI}dx$
Torsional Moment **T**	$\int_0^L \frac{T^2}{2GJ}dx$	$\int_0^L \frac{tT}{GJ}dx$

Ex: Determine the vertical displacement of joint *C* of the steel truss shown in fig. The cross-sectional area of each member is A = 400 mm^2 and E = 200 GPa.



Members	n	N	L	nNL
AB	0	100	4	0
BC	0	141.4	2.828	0
AC	-141.4	-141.4	2.828	565.429
CD	1	200	2	400

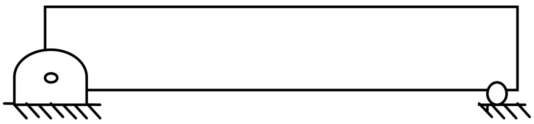
$\sum nNL = 965.7$ KNm

$1.\Delta = 965.7/200\times10^3$

$$1.\Delta = \frac{965.7}{200\times10^3\times400}$$

$\boxed{\Delta = 12.07\text{mm}}$

- **Methods of Virtual Forces Applied to Beam**



$$\boxed{1.\Delta = \int_0^L \frac{mM}{EI}dx}$$

M = Internal moment in the beam, expressed as function of *x* and caused by real loads.

E = Modulus of elasticity of the material.

I = Moment of Inertia of the X-sectional area, computed about neutral axis.

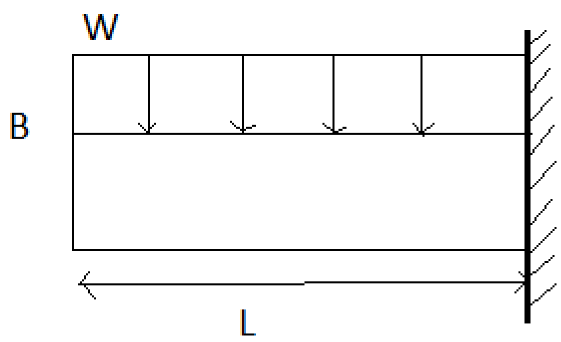
1 = External virtual load (unit) acting on the beam in the direction of Δ.

Δ = Displacement caused by the real loads acting on the beam.

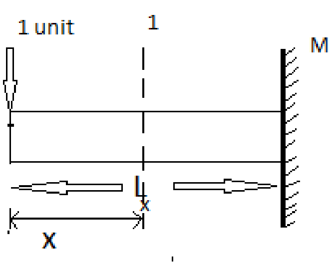
m = internal virtual moment in the exam expressed as a function of x and caused the external virtual unit load.

$$\boxed{1.\theta = \int_0^L \frac{m.M}{EI} dx}$$

Ex: Determine the displacement of B on the beam. E.I = constant



Sol:



m= (-1) x

$$\Delta = \int_0^L \frac{mM}{EI} dx = \int_0^L \frac{Wx^3}{2EI} dx$$

$$\Delta = \frac{wL^4}{8EI}$$

Ex: Determine the slope at B of a beam shown in the figure:-

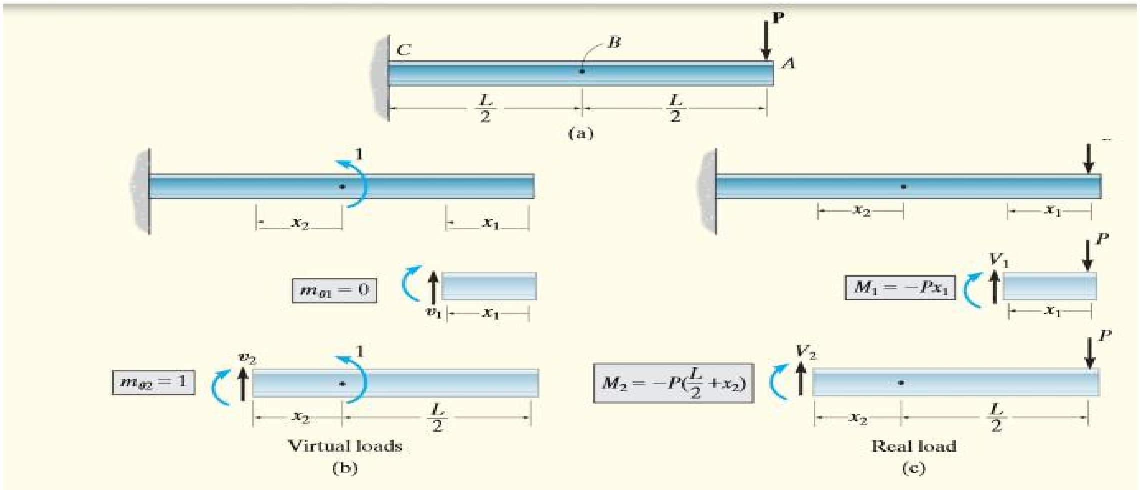


Fig. 14–37

Virtual-Work Equation, The slope at B is thus:

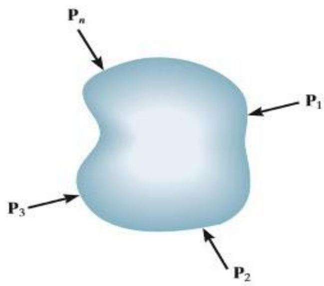
$$1.\theta = \int m_{\theta}.M/EI * dx$$

$$\frac{\int_0^{\frac{L}{2}} \cdot 0(-px)dx \cdot_1 .}{EI} + \frac{\int_0^{\frac{L}{2}} 1\left\{-P\left[\left(\frac{L}{2}\right)+X2\right]\right\}dx \cdot_2}{EI}$$

$$\theta_b = \frac{-3PL^2}{8EI}$$

The negative sign indicates that θ_b is clockwise, that is, opposite to the direction of the virtual couple moment shown in fig.

- **Castiglione's Theorem:**



$$U_i = U_e = f(P_1, P_2, P_3, \ldots P_i, P_n)$$

$$U_i + du_i = U_i + \partial u_i/\partial P_i\,(dP_i)$$

$$U_i + du_i = U_i + ½(dP_i d\Delta_i) + dP_i\Delta_i$$

$$\boxed{\Delta_i = \frac{\partial Ui}{\partial Pi}}$$

TRUSSES:

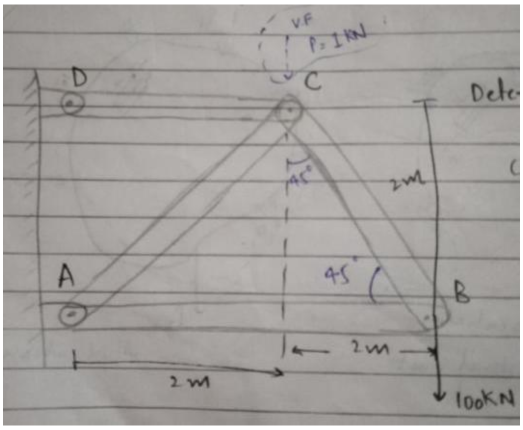
$$\Delta = \frac{\partial(\sum N^2 L)}{\partial P (2AE)}$$

$$\boxed{\boldsymbol{\Delta = \sum N \left(\frac{\partial N}{\partial P}\right) \frac{L}{AE}}}$$

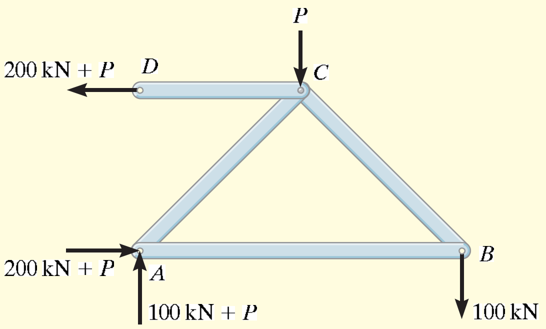
BEAMS:

$$\boxed{\boldsymbol{\Delta = \int_0^L M \left(\frac{\partial M}{\partial P}\right) \frac{dx}{EI}}}$$

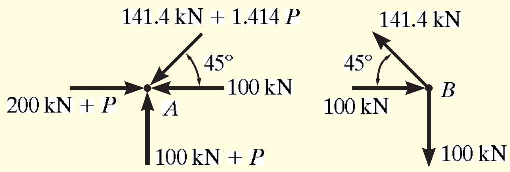
Ex: Determine the vertical displacement of joint *C* of the steel truss shown in fig. The cross-sectional area of each member is *A* = 400 mm^2 and *E* = 200 GPa.



Members	N	$N)_{p=0}$	L	$\frac{\partial N}{\partial P}$	$N)_{P=0}.\ L.\ \frac{\partial N}{\partial P}$
AB	-100	-100	4	0	0
BC	$-\sqrt{2}.100$	-141.4	2.828	0	0
AC	$(P+100)\sqrt{2}$	141.4	2.828	$\sqrt{2}$	565.429
CD	-200-P	-200	2	-1	400



(b)



(c)

Internal Forces *N*. The reactions at the truss supports A and D are calculated and the results are shown in Fig. . Using the method of joints, the N forces in each member are determined
For convenience, these results along with their partial derivatives $\partial N/\partial P$ are listed in tabular form. Note that since **P** does not actually exist as a real load on the truss, we require $P = 0$.

Member	N	$\dfrac{\partial N}{\partial P}$	$N(P = 0)$	L	$N\left(\dfrac{\partial N}{\partial P}\right)L$
AB	-100	0	-100	4	0
BC	141.4	0	141.4	2.828	0
AC	$-(141.4 + 1.414P)$	-1.414	-141.4	2.828	565.7
CD	$200 + P$	1	200	2	400
					Σ 965.7 kN · m

Castigliano's Second Theorem.

$$\Delta_{C_v} = \Sigma N\left(\frac{\partial N}{\partial P}\right)\frac{L}{AE} = \frac{965.7 \text{ kN}\cdot\text{m}}{AE}$$

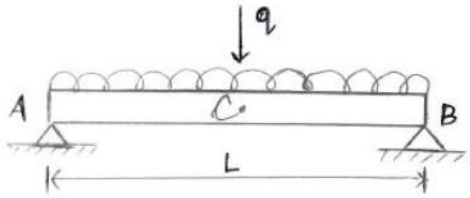
Substituting the numerical values for A and E, we get

$$\Delta_{C_v} = \frac{965.7 \text{ kN}\cdot\text{m}}{[400(10^{-6})\text{ m}^2]\,200(10^6)\text{ kN/m}^2}$$

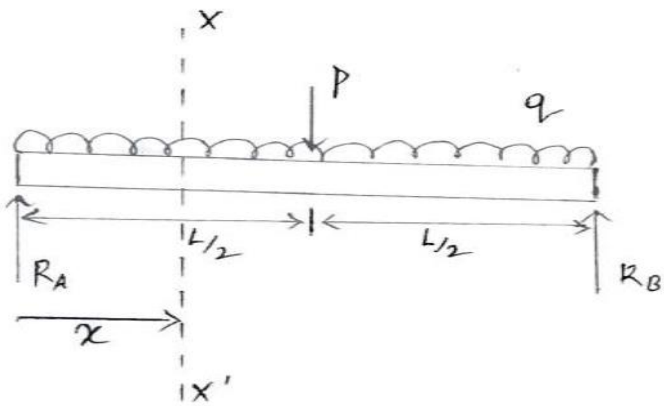
$$= 0.01207 \text{ m} = 12.1 \text{ mm} \qquad \textit{Ans.}$$

Ex: A simply supported beam is loaded with a UDL ‘q’ per unit length over the span ‘L’. Using Cast. Theorem, Find

i. Deflection at mid span
ii. Slope at one end



Sol:



$$R_A + R_B = P + qL$$

$$\sum \mathrm{M_B} = 0$$

$$\mathrm{P} \times \frac{\mathrm{L}}{2} + \mathrm{q}\frac{\mathrm{L}^2}{2} - R_A(L) = 0$$

$$\boxed{\mathrm{R_A} = \left(\frac{\mathrm{qL + p}}{2}\right)}$$

$$\boxed{\mathrm{R_B} = \left(\frac{\mathrm{qL + p}}{2}\right)}$$

$M_x = R_A.x – qx^2/2$

$M_x = (qL + P).X/2 - qx^2/2$

$\frac{dMx}{dp}$ = x/2 [for 0 < x < L/2]

$U = {}_0^L\int M_x^2 . dx/2EI$

$\Delta_c = \frac{dU}{dP} = 1/EI{}_0^L\int M_x\,(\partial M_x/\partial p).dx$

$M_x = (qL + P).X/2 - qx^2/2 - P(x\text{-}L/2)$ for $(L/2<x<L)$

$\frac{dMx}{dp} = L/2 - x/2$

$\Delta_c = \frac{dU}{dP}$

$\Delta_c = \frac{1}{2EI}\left[\int_0^{l/2}(ql\,x^2 - qx^3)dx\right]$

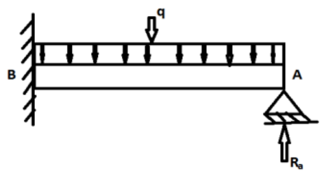
$= \frac{1}{2EI}\left[(qlx^3/3)-(qx^4/4)]\right)dx\right] = \frac{1}{24EI}\left[4qLx^3-3qx^4\right]$

$= \frac{1}{24EI}\left[4qL.L^3/8-3qL^4/16\right]$

$= \frac{1}{24EI}\left[2qL^4-3qL^4\right]$

$\mathbf{\Delta_c = \frac{5ql^4}{384}}$ ans.

Ex:

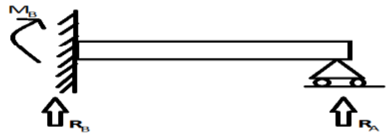


$$\Delta_A = \frac{1}{EI}\int_0^L M_X\left(\frac{\partial M_X}{\partial X}\right)dx = 0$$

$$M_X = X_X - \frac{qx^2}{2}$$

$$0 = \frac{1}{EI}\int_0^L\left(X_X - \frac{qx^2}{2}\right)xdx$$

$$\mathbf{X = \frac{3}{8}ql}\ \text{ans.}$$



$$R_A + R_B = ql \ ; \quad R_B = \frac{5}{8}\,ql$$

$$\sum M_A = 0$$

$$R_B \times L + M_b - \frac{ql^2}{2} = 0$$

$$\mathbf{M_b = -\frac{1}{8}ql^2} \text{ ans.}$$

- **MAXWELL'S RECIPROCAL THEOREM**

Maxwell's reciprocal work theorem states that for a linear elastic structure subjected to two set of forces **P** and **Q**, the work done by the set **P** through the displacements produced by the set **Q** is equal to the work done by the set **Q** through the displacements produced by the set **P**.

This theorem has applications in structural engineering where it is used to define influence lines and derive the boundary element method.

PROOF:

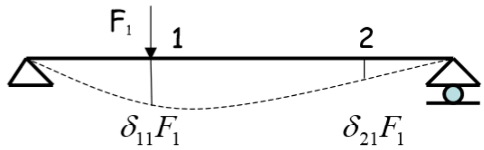


Fig. 1

Let us consider the beam in the figure. Because of the load F_1 the beam deflects an amount $\delta_{11}F_1$ at point 1 and amount $\delta_{21}F_1$ at point 2.

Where δ_{11} and δ_{21} are the deflections at points 1 and 2 due to a unit load at point 1.

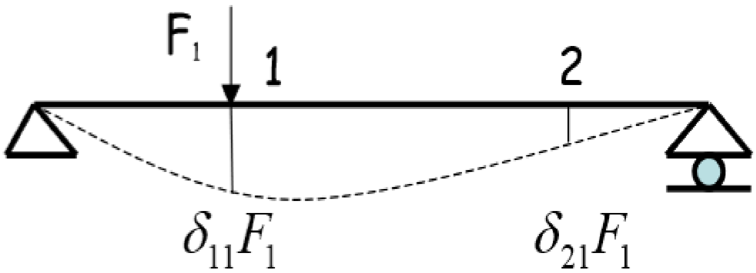


Fig. 2

δ_{ij} deflection at point **i** due to a unit load at **j**.

i =place of deflection.

j= place of unit load.

Now we will formulate an expression for the work due to F_1 and F_2.

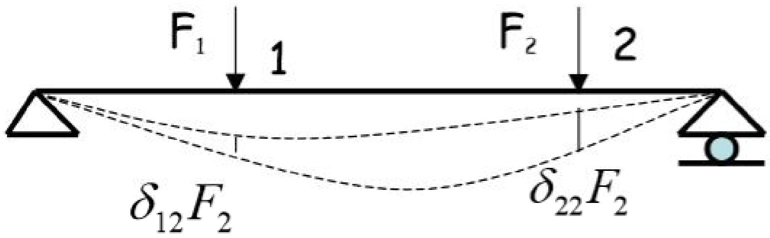


Fig. 3

Apply the forces F_1 and F_2 simultaneously the resulting work can be written

$$\mathbf{W = ½ (F_1\Delta_1 + F_2\Delta_2)}$$

$$\mathbf{W = ½ (F_1(\delta_{11}F_1 + \delta_{12}F_2) + F_2(\delta_{21}F_1 + \delta_{22}F_2))}$$

$$\mathbf{W = ½ (\delta_{11}F_1^2 + (\delta_{12} + \delta_{21})\, F_1F_2 + \delta_{22}F_2^2) \quad .... (1)}$$

$\Delta_1 = \delta_{11}F_1 + \delta_{12}F_2$ $\Delta_2 = \delta_{21}F_1 + \delta_{22}F_2$

Now if we apply forces one by one, we get

If we apply F_1 first the amount of work performed is

$$W_1 = ½(F_1\ \delta_{11}F_1) = ½(\ \delta_{11}F_1^2)$$

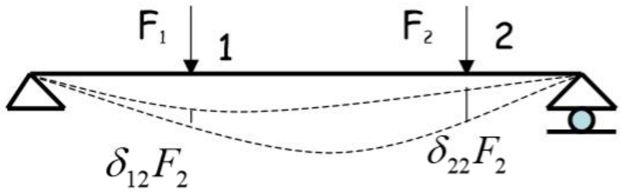


Fig. 4

Next, we apply F_2 to the beam on which F_1 is already acting.

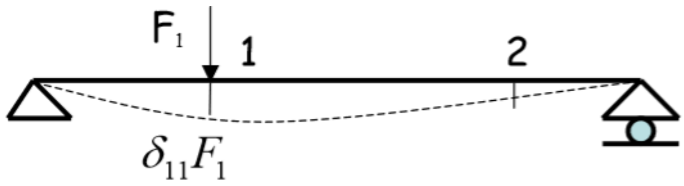


Fig. 5

The additional work resulting from the application of F_2

$$W_2 = F_1(\ \delta_{12}F_2) + ½(F_2(\delta_{22}F_2))$$

½ factor is absent on the first term because F_1 remains constant at its full value during the displacement. The total work due to F_1 and F_2

$$\mathbf{W = ½\ (\ \delta_{11}F_1^2 + \delta_{22}F_2^2) + \delta_{12}F_1F_2\ \ldots\ldots\ldots\ldots\ (2)}$$

In a linear system, the work performed by two forces is independent of the order in which the forces are applied. Hence the two works must be equal.

equating equations 1 and 2, we get,

$$\tfrac{1}{2}\left(\delta_{11}F_1^{\,2} + (\delta_{12} + \delta_{21})\, F_1F_2 + \delta_{22}F_2^{\,2}\right) = \tfrac{1}{2}\left(\delta_{11}F_1^{\,2} + \delta_{22}F_2^{\,2}\right) + \delta_{12}F_1F_2$$

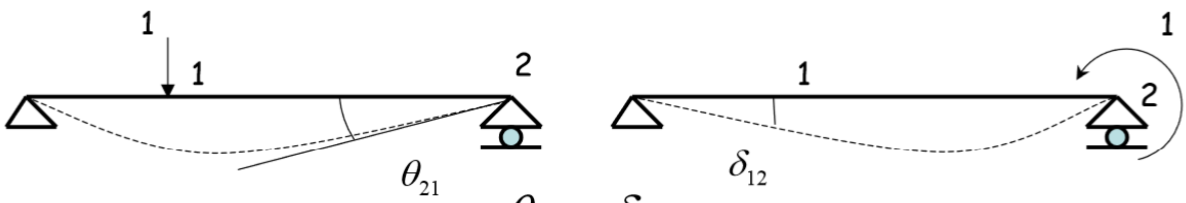
$$\tfrac{1}{2}(\delta_{12} + \delta_{21}) = \delta_{12}$$

$$\delta_{12} + \delta_{21} = 2\delta_{12}$$

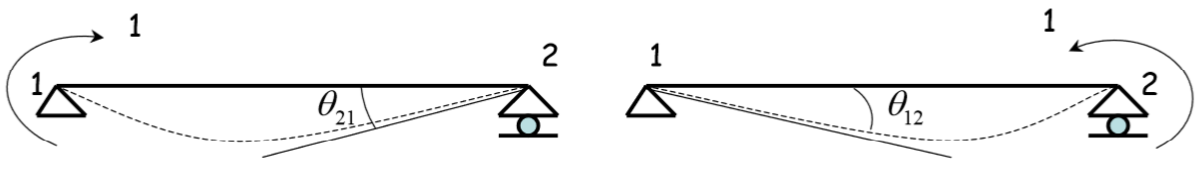
$$\boldsymbol{\delta_{12} = \delta_{21}}$$

This relationship is known as Maxwell's reciprocal theorem.

$$\delta_{21} = \delta_{12}$$

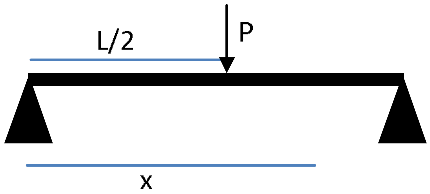


$$\theta_{21} = \delta_{12}$$



$$\theta_{21} = \theta_{12}$$

Ex:

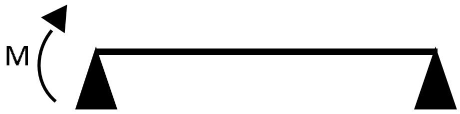


Sol:

$$R_a + R_b = P$$

$$\sum M_b = R_a * \frac{L}{2} - \frac{PL}{2} = 0$$

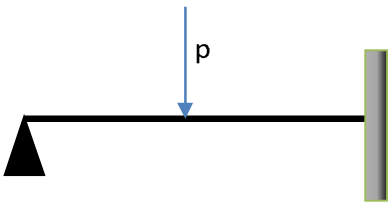
$$R_a = \frac{P}{2}$$

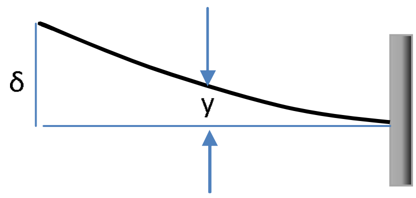


$$\boldsymbol{\theta_a = \frac{Pl^3}{16 \in I}}$$

$$\boldsymbol{\delta_c = \frac{ML^2}{16\epsilon I}}$$

Ex:





$$y = \frac{1}{6\epsilon I}(l - x)^2(2l + x)$$

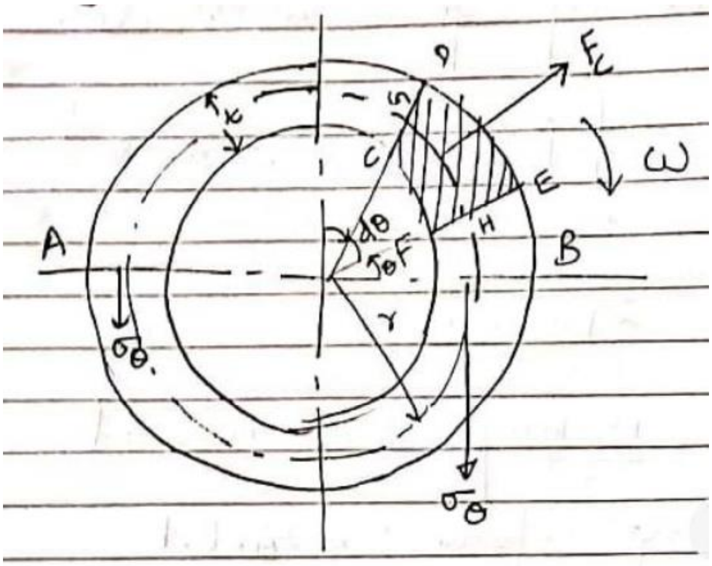
$$y_{Max} = \delta \text{ at } x = 0$$

Now acc. to reciprocal theorem

$$X.\delta - Py = 1.0\,;\ \mathbf{X}' = \frac{\mathbf{PY}}{\boldsymbol{\delta}}$$

$$\mathbf{X} = \frac{\sum \mathbf{p_n y_n}}{\boldsymbol{\delta}}$$

- **ROTATING RING**



Consider a thin ring rotating about its centre of gravity at “o” as shown.

ρ= density of the ring.

r= mean radius of the ring.

ω= angular velocity of rotation.

t= thickness.

Volume of element per unit length= r.dΘ.t

Centrifugal force acting on this element:

$$dF_c = \rho.r.d\Theta.t.\omega^2 r$$

Vertical component of this force = $dF_c \sin\Theta$.

Total bursting force across dia AB = $\int_0^{\pi} \rho.r.d\theta.t.\omega^2.r\ \sin\theta\, d\theta = 2\ \rho\omega^2 r^2 t$

If σ_θ is hoop stress induced,

Resisting force = 2. σ_θ.t.1

For equilibrium,

$$2\ \rho\omega^2 r^2 t = 2\sigma_\theta t$$

$$\boxed{\boldsymbol{\sigma_\theta = \rho\omega^2 r^2 = \rho v^2}}$$

Ex: The thin rim of a wheel is 90 cm diameter. Neglect the effect of spokes. How many revolutions per min may it make without the hoop stress exceeding 140 MPa. The density is 7800 kg/m^3. E=200 GPa also find change in diameter.

Sol: (i) r = d/2 = 45 cm = 0.45 m

$\sigma_\theta \leq$ 140 MPa

$\rho\omega^2 r^2 < 140 \times 10^6$ Pa

7800 Kg/m^3. $(\omega^2).(.45)^2$ m^2 < 140×10^6 Pa

$\omega^{2=}$ 8.8×10^4

$\omega = 2.976 \times 10^2$ rad/s

Now, $\omega = 2\pi N/60$

N = 28.43×10^2 rev/min

(ii) $\sigma_\theta = E\epsilon_\theta$

$\epsilon_\theta = 140\times10^6$ Pa / 200×10^9 Pa

$= 0.7 \times 10^{-3}$

Now,

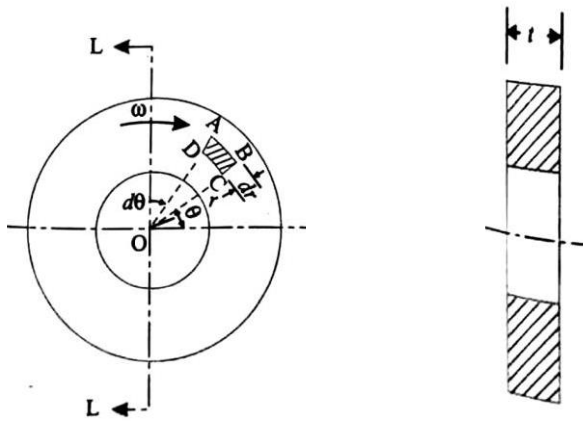
$\epsilon_\theta = \Delta d/d$

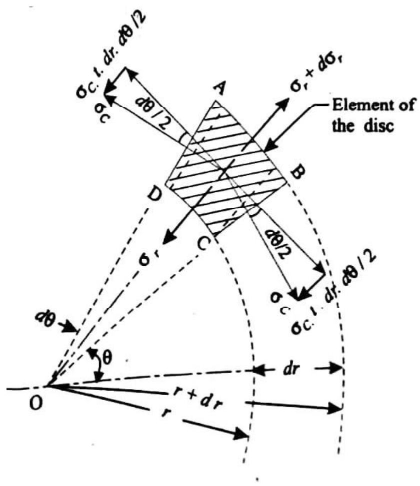
$\Delta d = 0.9\times0.7\times10^{-3}$ m

$\Delta d = 0.63 \times 10^{-3}$ m

Δd = 0.63 mm

- **ROTATING DISC**





Resolving Force in the radial outward direction: -

$$(\sigma+d\sigma_r)(r+dr)(d\theta)t \;-\; (\sigma_r)r.d\theta.t \;-\; \sigma_{\theta.}.t.dr.d\theta$$

Simplifying and neglecting small quantities;

$$(\sigma_r - \sigma_\theta)dr.d\theta.\,t \;+\; d\sigma_r.rd\theta.t$$

For equilibrium of the element

$$\rho r.d\theta.dr.t.\omega^2 \;+\; (\sigma_r - \sigma_\theta)dr.d\theta.t \;+\; d\sigma_r.rd\theta.t \;=\; 0$$

$$\rho\omega^2 r^2 \;+\; (\sigma_r - \sigma_\theta) \;+\; r(d\sigma_r/d\sigma_\theta) \;=\; 0$$

$$(\sigma_r - \sigma_\theta) \;=\; -\,r(d\sigma_r/d\sigma_\theta) \;-\; \rho\omega^2 r^2 \quad \text{----(a)}$$

On account of rotation $r \rightarrow r+n$

And $r+dr \rightarrow r+dr+dn$

Then the circumferential strain : $\epsilon_\theta = [2\pi(r+n) - 2\pi r] / 2\pi r = n/r$

and radial strain : $\epsilon_r = [(r+dr+dn) - (r+dr)] / dr$

$\epsilon_r = dn/dr$

$\epsilon_\theta = [1/E][\sigma_\theta - \nu\sigma_r] = n/2$

$Þ\omega^2\gamma^2 + (\sigma_r - \sigma_\Theta) + \gamma d\sigma_\gamma/d\gamma = 0$

$\sigma_\gamma - \sigma_\Theta = -\gamma d\sigma_\gamma/d\gamma - þ\gamma\omega^2\gamma^2$ ----(a)

on account of rotation $\gamma \to \gamma+u$

& $\gamma+d\gamma \to \gamma+d\gamma+du$

Then circumsential strain $\epsilon_{o} =$ $(2\pi(\gamma+u) - 2\pi\gamma)/2\pi\gamma$ **=** u/γ

& **radial strain** $\epsilon_r = ((\gamma+d\gamma+du)-(\gamma+d\gamma))/d\gamma = du/d\gamma$

$\epsilon_\Theta = [\sigma_\Theta - \nu\sigma_\gamma]/E = u/2$

$\epsilon_\gamma = [\sigma_\gamma - \mu\sigma_\Theta]/E = du/d\gamma$ ------- (c)

$u = \gamma[\sigma_\Theta - \mu\sigma_\gamma]/E$ ---(b)

Diff w.r.t. r

$du/dr = r/E(\frac{d\sigma}{dr} - \gamma d\sigma r/dr) + 1/E(\sigma\theta - \mu\sigma r)$ ....(d)

comparing (c) and (d)

$1/E(\sigma_r - \mu\sigma\theta) = r/E\,(\frac{d\sigma\theta}{dr} - \mu\frac{d\sigma r}{dr}) + 1/E\,(\sigma\theta - \mu\sigma r)$

$\Rightarrow \quad \sigma_r(1+\mu) - \sigma\theta(1+\mu) = r(\frac{d\sigma\theta}{dr} - \frac{\mu d\sigma r}{dr})$

$\Rightarrow \quad (1+\mu)(\sigma r - \sigma\theta) = r(\frac{d\sigma\theta}{dr} - \frac{\mu d\sigma r}{dr})$................(e)

Substituting (a) in (e)

$d/dr\,(\sigma r + \sigma\theta) + (1+\mu)(\rho\omega^2 r) = 0$............(f)

integrating, we get

$\sigma_r + \sigma\theta + 1/2(1+\vartheta)\rho\omega^2 r^2 = c_1$--------(g)

Where c_1 is the constant of integration

$\sigma\theta = c_1 - \sigma_r - 1/2(1+\vartheta)\rho\omega^2 r^2$

Substituting in (a)

$d/dr\,(r^2\sigma_r)=c_1 r-(3+\vartheta)\rho\omega^2 r^2/2$

Integrating, we get,

$$\# \ \sigma_r = \frac{C1}{2} + \frac{C2}{r^2} - \left(\frac{3+\mu}{8}\right)\rho\omega^2 r^2$$

$$\# \ \sigma_c = \frac{C1}{2} - \frac{C2}{r^2} - \left(\frac{1+3\mu}{8}\right)\rho\omega^2 r^2$$

Case 1: **Solid Disc**

At the centre r=0; Stresses cannot be infinite at the centre of disc; therefore

$$C_2 = 0.$$

$$\sigma_r = C_1/2 - \{(3+\mu)/8\}\,\rho\omega^2 r^2$$

$$\sigma_c = C_1/2 - \{(1+3\mu)/8\}\,\rho\omega^2 r^2$$

at $r=r_2$ (outer radius); $\sigma_r = 0$

$$C_1 = \{(3+\mu)/4\}\,\rho\omega^2\,(r_2)^2$$

$$\sigma_r = \left(\frac{3+\mu}{8}\right)\rho\omega^2\,\{(r_2)^2 - (r)^2\}$$

$$\sigma_c = ((3+\mu)/8)\rho\omega^2 r_2^{\,2} - (1+3\mu)/8\rho\omega^2 r_2^{\,2}$$

$$\sigma_c = \rho\omega^2/8[(3+\mu)r_2^{\,2} - (1+3\mu)r^2]$$

at r=r2

$$\sigma_c = \left(\frac{\rho\omega^2}{8}\right)[2\,r_2^{\,2} - 2\,r_2^{\,2}\mu]$$

Note:

$$(\sigma_c)_{at\,(r=r2)} = \frac{\rho\omega^2}{4}(1-\mu)r_2^{\,2}$$

At r = 0; σ_c, σ_r are max.

$$(\sigma_c)_{max} = (\sigma_r)_{max} = \left(\frac{3+\mu}{8}\right)\rho\omega^2 r^2$$

Case2: **Hollow disc**

$$\sigma_r=(c_1/2)+(c_2/r^2)-(3+\nu)\rho\omega^2r^2/8;$$

$$\sigma_c=(c_1/2)-(c_2/r^2)-(1+3\nu)\rho\omega^2r^2/8;$$

At $r=r_1$, $\sigma_r=0$ &

At $r=r_2$, $\sigma_r=0$

$$c_1=(3+\nu)\rho\omega^2(r_1^2+r_2^2)/4$$

$$c_2=-(3+\nu)\ \rho\omega^2r_1^2r_2^2/8$$

$$\# \ \sigma_r = \left(\frac{3+\mu}{8}\right)\rho\omega^2[r_1^2+r_2^2-(r_1^2r_2^2/r^2)-r^2]$$

$$\# \ \sigma_c= \left(\frac{3+\mu}{8}\right)\rho\omega^2[r_1^2+r_2^2+(r_1^2r_2^2/r^2)-\frac{1+3\nu}{3+\nu}r^2]$$

$$\sigma_c]_{max}=\left(\frac{3+\mu}{4}\right)\rho\omega^2[r_2^2+(1-\nu)r_1^2/(3+\nu)]$$

for $\sigma_r|_{max}$, $(d\sigma_r/dr)=0$

$$r = \sqrt{r1r2}$$

$$\sigma_r]_{max} = \left\{\frac{3+\mu}{8}\right\}\rho\omega^2(r_2-r_1)^2$$

Note: Hollow disc with pin hole at the centre

As, $r_1 \rightarrow 0$

$$\sigma_c]_{max} = \left(\frac{3+\mu}{8}\right)\rho\omega^2r_2^2$$

when $r_1 \rightarrow r_2 \rightarrow r$

$$\sigma_c]_{max} = \rho\omega^2r^2$$

Ex: Determine the intensities of principle stresses in flat steel disc of uniform thickness having a diameter of 1m & rotating at 2400 rpm. What will be the stresses, if the disc has a central hole of 0.2m diameter?

Poisson's ratio= 1/3 & ρ=7850kg/m^3

Sol:

ω (rad/s)= 2πN/60 = 2π*2400/60 = 80π = 251.2 rad/s

Case 1:   Solid disc

$$\sigma_r = \frac{C_1}{2} + \frac{C_2}{r^2} - \frac{(3+\mu)\rho\omega^2 r^2}{8}$$

$$\sigma_c = \frac{C_1}{2} - \frac{C_2}{r^2} - \frac{(1+3\mu)\rho\omega^2 r^2}{8}$$

$$\sigma_{r\ \rightarrow} \infty \text{ at } r = 0$$

$$C_2 = 0$$

$$r = 0.5; \sigma_r = 0$$

$$0 = \frac{C_1}{2} - \frac{\left(3+\frac{1}{5}\right)(7850)(251.2)^2(0.5)^2}{8}$$

$$\mathbf{C_1 = 103.3\frac{MN}{m^2}}$$

$$\sigma_r = 103.3\frac{MN}{m^2} - \frac{\left(3+\frac{1}{3}\right)\rho\omega^2 r^2}{8}$$

$$\text{at } r = 0.1 \text{ m}$$

$$\sigma_r = 51.65\frac{MN}{m^2} - \left(\frac{3.33}{8}\right)(7850)(251.2)^2(0.1)^2$$

$$\sigma_r = 51.65\frac{MN}{m^2} - 2.06 \times \omega\frac{MN}{m^2}$$

$$\boldsymbol{\sigma_r = 49.59\frac{MN}{m^2}}$$

$$\text{at } r = 0$$

$$\boldsymbol{\sigma_r = (\sigma_r)\mathbf{max} = 51.65\frac{MN}{m^2} = \sigma_c}$$

$$\text{at } r = 0.5\ \sigma_r = 0$$

Case 2 : Hollow disc

r_1 = 0.1m r_2=0.5m

$$\sigma_r = 0 \text{ at } r = r_1;\ r_2$$

$$0 = \frac{C_1}{2} + \frac{C_2}{(0.1)^2} - \left(\frac{3+\mu}{8}\right)\rho\omega^2(0.1)^2$$

$$0 = \frac{C_1}{2} + \frac{C_2}{(0.5)^2} - \left(\frac{3+\mu}{8}\right)\rho\omega^2(0.5)^2$$

OR

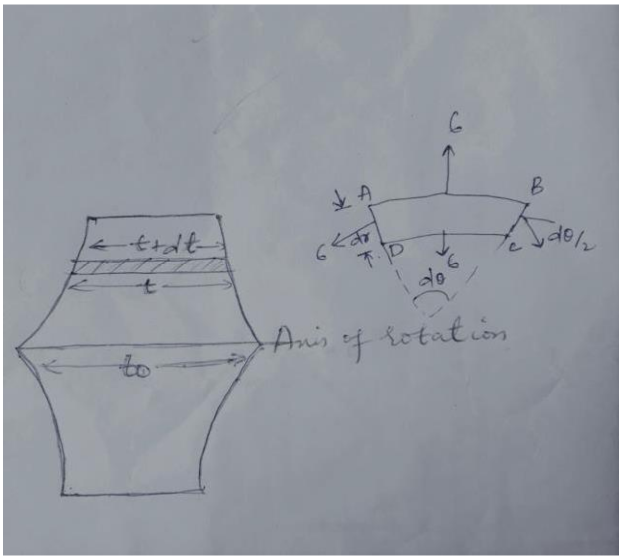
$$C_1 = \left(\frac{3+\frac{1}{3}}{4}\right)(7850)(251.2)^2(0.1^2+0.5^2)$$

$$\mathbf{C_1 = 107.21 MN/m^2}$$

$$C_2 = -\left(\frac{3+\frac{1}{3}}{4}\right)(7850)(251.2)^2(0.1)^2(0.5^2)$$

$$\mathbf{C_2 = -\ 0.515 MN/m^2}$$

- **DISC OF UNIFORM STRENGTH**



σ = uniform stress in radial and circumferential direction.

Volume of element = rdθ.t.dr

Centrifugal force acting on the element ABCD due to rotation $= m\omega^2 r$

$= (\rho r d\theta.t.dr)\omega^2 r$

$= \rho d\theta.t.dr.\omega^2 r^2$

Radial force on face DC = rdθ.t.σ

Radial force on face AB=(r+dr).dϴ.(t+dt)σ

Centrifugal forces on BC and DA=t.dr.σ

Resolving forces in radial direction,

$\rho d\theta t.dr\omega^2r^2$ + $(r+dr)\theta(t+dt)\sigma$ = $rd\theta.t.\sigma$ + $2\,\sigma\, t.dr.\sin d\theta/2$

=> $t\rho dr\omega^2r^2 + rt\sigma + rdt\sigma + drt\sigma = rt\sigma + tdr\sigma$

=> $t\rho dr\omega^2r^2 + rdt\sigma = 0$

or $\sigma dt/t = -\rho\omega^2 rdr$

$\ln t = \ln A - \rho\omega^2r^2/2\sigma$

Or $t/A = e^{-\rho\omega^2r^2/2\sigma}$

At $r=0$; $t=t_0$

$$\boxed{\boldsymbol{t = t_0\, e^{-\rho\omega^2r^2/2\sigma}}}$$

Ex: Steam turbine is designed such that the radial and circumferential stresses are constant throughout and are equal to 90N/mm^2, when running at 400 rpm. If the axial thickness at centre =20mm, what is the thickness at r=400mm. Assume density of rotor =7800kg/m^3.

Sol:

σ = 90N/mm^2 = 90MPa

$\omega = 2\pi/60 \times 4000 = 418.67$ rad/s

t_0= 20mm = .02m

t=? At r=400mm = .4m

$t = t_0 e^{(-\rho\omega 2r2/2\sigma)}$

Now X= (7800kg/m^2)(418.67)^2(0.4)^2 / 2×90×10^6

X = 1.215

t= (.02) ($e^{-1.215}$)

t= 5.93 mm ans.

- **ROTATING LONG CYLINDERS**

$\varepsilon_r = [\sigma_r - \nu(\sigma_l + \sigma_c)]/E = du/dr$ ……….(1)

$\varepsilon_c = [\sigma_c - \nu(\sigma_r + \sigma_l)]/E = u/r$ …………..(2)

$\varepsilon_l = [\sigma_l - \nu(\sigma_r + \sigma_c)]/E$ …………(3)

from eq. (2)

$E\mu = r [\sigma_c - \nu (\sigma_r + \sigma_l)]$

Diff. with respect to 'r'

$(\sigma_r - \sigma_c)(1+\mu) = r [(d\sigma_c/dr) - \mu((d\sigma_c/dr) + (d\sigma_l/dr))]$ ------ (4)

From eq. (2)

$E\epsilon_l = \sigma_l - \mu(\sigma_r + \sigma_c) = C_1$

C_1 = constant

$\sigma = C_1 + \mu(\sigma_r + \sigma_c)$

diff. with respect to 'r'

$d\sigma l/dr = \mu[(d\sigma_r/dr) + (d\sigma_c/dr)]$ ---- (a)

eq. (a) substitute in eq. (4)

$(\sigma_r - \sigma_c) = r[(1-\mu)(d\sigma_r/dr) - \mu(d\sigma_r/dr)]$ -----(5)

Also, $(\sigma_r - \sigma_c) = (r(d\sigma_r/dr) + \rho\omega^2 r^2)$ ----- (6)

$\frac{d}{dr}(\sigma_r + \sigma_c) = \frac{-\rho}{1-\mu}\omega^2 r$

Integrating...

$\sigma_r + \sigma_c = \frac{-\rho}{1-\mu}\omega^2\frac{r^2}{2} + c_2$...eqn 7

Adding eqn 6 and eqn 7...

$2\sigma_r + r\frac{d}{dr}\sigma_r = = \frac{-\rho\omega^2}{2} r^2 \frac{3-2\mu}{2(1-2\mu)} + c_2$

Multiply both the side by "r"

$$\frac{d}{dr}(r^2\sigma_r) = \frac{-\rho\omega^2 r^3}{2}.\left(\frac{3-2\mu}{1-\mu}\right) + rc_2$$

Integrating...

$$\# \ \boldsymbol{\sigma_r = \frac{c_2}{2} + \frac{c_3}{r^2} - \frac{\rho}{8}\,\omega^2 r^2 \left(\frac{3-2\mu}{1-\mu}\right)}$$

Substituting in eqn 7...

$$\# \ \boldsymbol{\sigma_c = \frac{c_2}{2} - \frac{c_3}{r^2} - \frac{\rho}{8}\,\omega^2 r^2 \left(\frac{1+2\mu}{1-\mu}\right)}$$

Case I: **Solid Cylinder**

$\boldsymbol{\sigma_r \Rightarrow \infty}$ **&** $\boldsymbol{\sigma_c \Rightarrow \infty}$ at the centre i.e. r = 0

Hence, $\boldsymbol{C_3 = 0}$, therefore,

$$\sigma_r = \frac{C2}{2} - \frac{\rho}{8}\,\omega^2 r^2 \left(\frac{3-2\mu}{1-\mu}\right)$$

$$\sigma_c = \frac{C2}{2} - \frac{\rho}{8}\,\omega^2 r^2 \left(\frac{1+2\mu}{1-\mu}\right)$$

$r = r_2$; $\sigma_r = 0$

$$\frac{C2}{2} = \frac{\rho}{8}\,\omega^2 r^2 \left(\frac{3-2\mu}{1-\mu}\right)$$

Thus,

$$\# \ \boldsymbol{\sigma_r = \rho\omega^2(r_2^2 - r^2)\left(\frac{3-2\mu}{8(1-\mu)}\right)}$$

$$\# \ \boldsymbol{\sigma_c = \rho\omega^2\left(\frac{3-2\mu}{8(1-\mu)}\right)\left[r_2^2 - \left(\frac{1+2\mu}{3-2\mu}\right)r^2\right]}$$

Note: Maximum stress occurs at the centre of the cylinder, where r = 0

Therefore, $\boldsymbol{\sigma_{r]max} = \sigma_{c]max} = \frac{\rho}{8}\,\omega^2 r^2{}_2 \left(\frac{3-2\mu}{1-\mu}\right)}$

Case 2: For **hollow cylinder**

$$\sigma_r = \frac{c_2}{2} + \frac{c_3}{r^2} - \frac{\rho}{8}\,\omega^2 r^2 \left(\frac{3-2\mu}{1-\mu}\right)$$

$r = r_1$; $\sigma_r = 0$

$r = r_2$; $\sigma_r = 0$

$$C_3 = -\frac{\rho}{8}\omega^2 r^2{}_2\ \ r^2{}_1\ \left(\frac{3-2\mu}{1-\mu}\right)$$

$$C_2/2 = \frac{\rho}{8}\,\omega^2\left(\frac{3-2\mu}{1-\mu}\right)[(r_1)^2 + (r_2)^2]$$

$$\#\ \boldsymbol{\sigma_r = \frac{1}{8}\rho\omega^2\left(\frac{3-2\mu}{1-\mu}\right)[r_1^2 + r_2^2 - (r_1^2 r_2^2/r^2) - r^2]}$$

$$\#\ \boldsymbol{\sigma_c = \frac{1}{8}\rho\omega^2\left(\frac{3-2\mu}{1-\mu}\right)[r_1^2 + r_2^2 + (r_1^2 r_2^2/r^2) - \frac{1+2\nu}{3-2\nu}r^2]}$$

$\sigma_r \rightarrow \sigma_r]_{\text{Max}}$ At **r** = $\sqrt{\mathbf{r_1 . r_2}}$

$$\boldsymbol{\sigma_r]_{Max}} = \frac{\boldsymbol{\rho\omega^2}}{\mathbf{8}}\langle\frac{\mathbf{3-2\mu}}{\mathbf{1-\mu}}\rangle\,(\mathbf{r_1} - \ \mathbf{r}_2)^2$$

$$\sigma_c]_{\text{Max}}\ \ \text{at}\, r = \ r_1$$

$$\boldsymbol{\sigma_c]_{Max}} = \frac{\boldsymbol{\rho\omega^2}}{\mathbf{8}}\langle\frac{\mathbf{3-2\mu}}{\mathbf{1-\mu}}\rangle\,[(\mathbf{2r_2}^2 + \mathbf{r_1}^2) - \left(\frac{\mathbf{1+2\mu}}{\mathbf{3-2\mu}}\right)\mathbf{r_1}^2]$$

Ex: A long cylinder of radius 300mm is rotating at 4500 rpm. The density of the material is 7800 Kg/m^3 & μ=0.3. Calculate the maximum stress in cylinder & draw the variation of σ_r & σ_c along the radius.

Sol:

$$\sigma_r]_{\text{Max}} = \sigma_c]_{\text{Max}} = \frac{\rho(r\omega)^2}{8}\langle\frac{3-2\mu}{1-\mu}\rangle$$

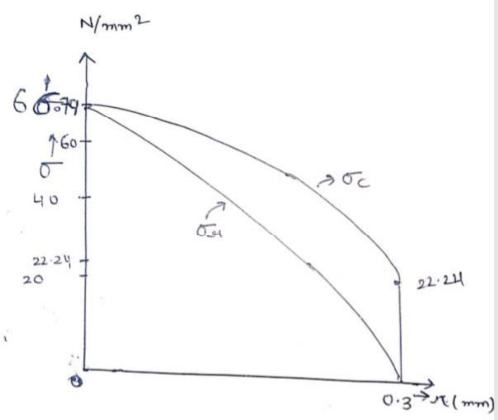
$$\frac{7800 * (\frac{4500 * 2\pi * 0.3}{60})^2}{8}\langle\frac{3-2*0.3}{1-0.3}\rangle$$

On solving we get

$$\boldsymbol{\sigma_r]_{Max} = \sigma_c]_{Max} = 66.74\ MN/m^2}$$

At r=0.3 ; $\sigma_r = \frac{\rho\omega^2}{8}\ (0.3^2 - \ 0.3^2) = 0$

$$\boldsymbol{\sigma_c} \ \ = \ \frac{\rho\omega^2}{8}\langle\frac{3-2\mu}{1-\mu}\rangle\left[r_2{}^2 - \frac{1+2\mu}{3-2\mu}\right]r_1{}^2 = \mathbf{22.24}\ \text{MN/m}^2$$



Ex: Hollow cylinder 20mm external radius and 100mm internal radius is rotating at 300rpm , density(ρ) = 7800 kg/m^3 and poisson's ratio (μ) = 0.3 ; Calculate maximum stress in cylinder . Plot variation of radial and hoop stresses?

Sol:

Given: r_1= 0.1m; r_2= 0.2m

(1) $\sigma_r)_{max}$

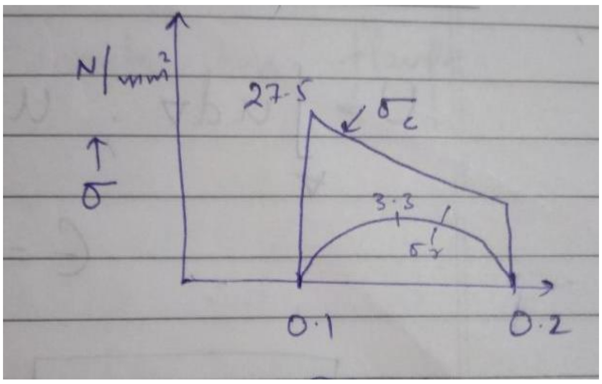
At r = $\sqrt{r1r2}$ = 0.1414m

σ_r)max = $\frac{\rho\omega^2}{8}(\frac{3-2(0.3)}{1-(0.3)})(0.2-0.1)^2$

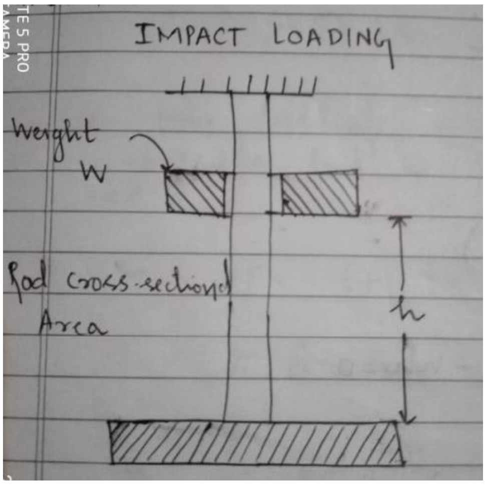
σ_r)max = 3.3 N/mm^2

(2) $\sigma_c)_{max}$ = $\frac{\rho\omega^2}{8}(\frac{3-2\mu}{1-\mu})[(2{r_2}^2+{r_1}^2)-(\frac{1+2\mu}{3-2\mu}){r_1}^2]$

σ_c)max = 27.5 N/mm^2



- **IMPACT LOADING**



A prismatic linear elastic of crossectional area A and length L (negligible mass) is hanging freely ,when a rigid weight W is allowed to fall freely from a height on the rod as shown in fig. end of the rod contains flanges (wt. = 0kg). Assuming no energy loss, calculate the longitudinal extension of rod due to impact.

Let us assume that rod deflects by an amount delta (S)

Considering conservation energy –

Initial PE of weight = strain energy stored with the elastic rod

$$W(h+ \delta)=u$$

$U=\int u\,dv,$ $\quad$ $u= \frac{\sigma^2}{2E} = E\,\varepsilon^2/2$

$$\varepsilon =(\delta/L)$$

$$U = \frac{AE\ \delta\text{^}2}{2L}$$

$$\frac{AE}{2L}\ \delta^2 = W(h+ \delta)$$

$$\frac{AE}{2L}\ \delta^2 - W\ \delta - Wh=0$$

$$\delta = \frac{1}{AE/L}\ (W \pm \sqrt{W^2 + 2EAWh/L})$$

$$\boldsymbol{\delta = \frac{WL}{EA}[1+\sqrt{1+\frac{2EAh}{WL}}\,]}$$

In the above equation $\delta st = \frac{WL}{EA}$ is the static deflection or elongation of rod when W is applied statiscally the term within the bracket denotes dynamic amplification factor (DAF) which when applied by static deflection gives dynamic elongation of rod

2. In the above expression, if h=0 i.e. load is suddenly applied to the flange of rod.

We get,

DAF=2 &

$\delta = 2\delta st$

i.e. under suddenly applied force, dynamic deflection load is twice of static application load.

$$\mathbf{DAF = 1+\sqrt{1+\frac{AEV^2}{WLg}}}$$

The stress developed due to this elongation assuming to be in the elastic limit.

$\sigma = E\varepsilon = E(\delta/L)$

$$\delta = \frac{W}{A}[1+\sqrt{1+\frac{2EAh}{WL}}]$$

 # $\sigma = \sigma_{st}$.DAF

 # $\varepsilon = \varepsilon_{st}$.DAF

Ex: Object of weight w is dropped over the middle of a simply supported beam from a height 'h'. The beam has cross-section 'A'. If h>> δst. Also keeping the mass of beam very small obtain the expressions for max bending stresses due to following wt.

$$\delta_{st} = WL^3/48EI$$

sol: P.E. of W = S.E. due to bending

$W(h+\delta) = K\delta^2/2$

$K = W/\delta st = 48EI/L^3$

So $K\delta^2 - 2W\delta - 2Wh = 0$

$\delta^2 - 2\delta W/K - 2Wh)K = 0$

$\delta^2 - 2\delta\ \delta st - 2\ \delta st\ h = 0$

$\boldsymbol{\delta = \delta st[1+(1+2h/\delta st)^{\wedge}½]}$

$D\ A\ F = 1 + \sqrt{1 + 2h/\delta st}$

$D\ A\ F = 1 + \sqrt{(2h/\delta st)(1 + \delta st\ /2h)½}$

$D\ A\ F = 1 + \sqrt{2h/\delta_{st}}$ [for $h >> \delta_{st}$]

$\sigma = (\sigma_{st})_{max}\ (1 + \sqrt{2h/\delta st}\,)$

$\sigma_{st})_{max} = 16M/bh^2$

$\sigma_{st})_{max} = 3WL/2Ah^3$

Where $h \rightarrow$ depth & $b \rightarrow$ width

$M_{max} = WL^3/4\epsilon Ah^2$

And $\delta_{st} = WL^3/48\epsilon I$

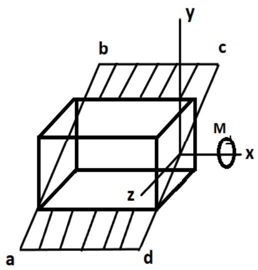
$\delta_{st} = WL^3/4\epsilon bh^3$

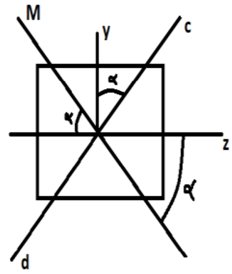
$\delta_{st} = WL^3/4\epsilon Ah^2$

$\boldsymbol{\sigma_{max} = \sqrt{18W\epsilon h/AL}}$

- **UNSYMMETRIC BEAM BENDING**

Bending about both principal axis:





Assuming elastic behaviour of the matiral super-position of stresses caused by m_y and m_z is the solution of the problem

$$\boldsymbol{\sigma_x = -M_zY/I_Z + M_YZ/I_Y}$$

$$\boldsymbol{\tan\beta = I_Z/I_Y\ (\tan\alpha)}$$

for neutral axis: σ_x=0

$$\frac{MzY}{Iz}=\frac{MyZ}{Iy}$$

$$\frac{Y}{Z}=\frac{My}{MZ}\times\frac{Iz}{Iy}$$

$$\frac{Y}{Z}=\tan\beta$$

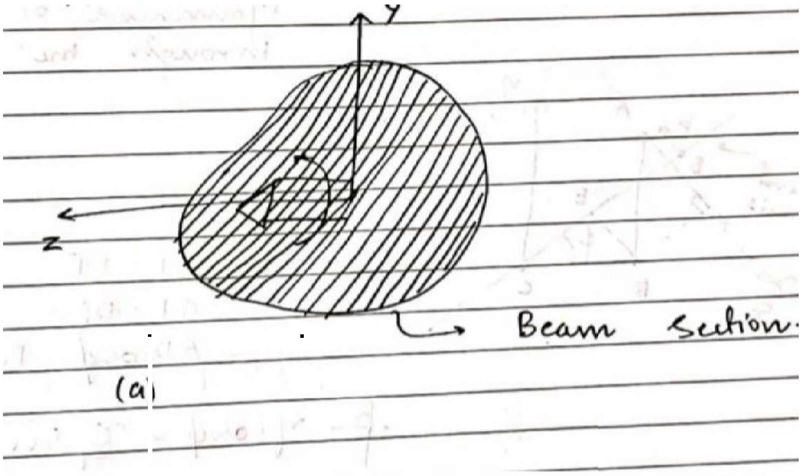
$$\frac{My}{Mz}=\sin\alpha$$

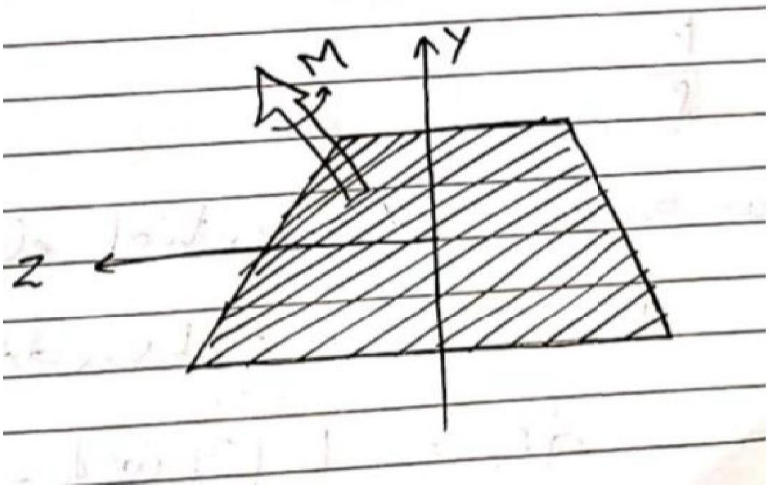
$$\boldsymbol{\tan\beta=\frac{Iz}{Iy}\tan\alpha}$$

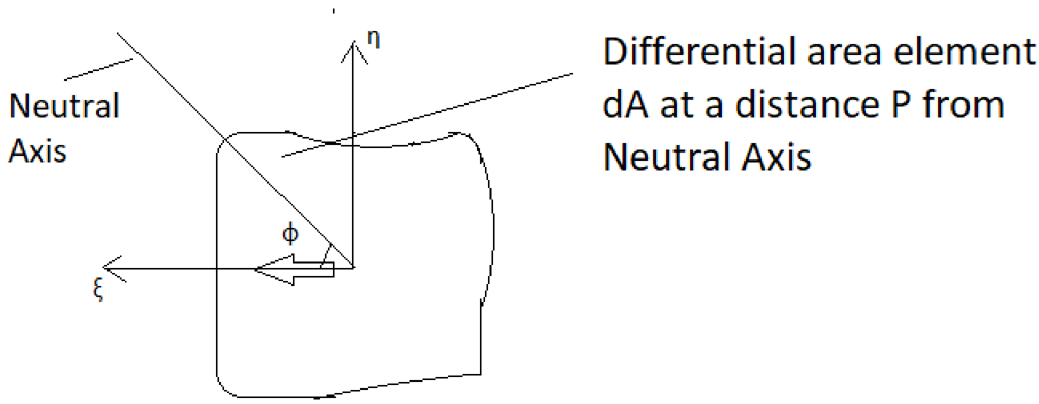
$\boldsymbol{\beta}$: angle made by NA with Z axis

- **Unsymmetric Bending of beam**

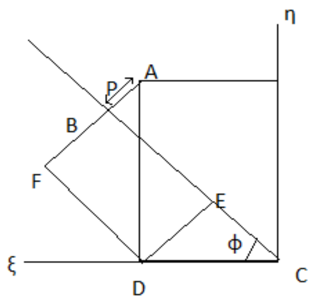
1. The beam does not possess any plane of symmetry.
2. The beam posses a plane of symmetry but that axis of bending moment is not normal to the plane of symmetry.







η,ξ are the principal axis of area moment of inertia through the centroid.



$P = AB = AF - BF$

$P = AF - DE$

$P = AD\cos\phi - DC\sin\phi$

$P = \eta\cos\phi - \xi\sin\phi$

$\epsilon_{xx} = -P/\rho$

Axial force on the differential element area is ;

$dF_x = \sigma_{xx} dA = E\epsilon_{xx} dA$

$dF_x = -E(\eta\cos\phi - \xi\sin\phi)dA/\rho$

Integrating, we get the resultant force N_x acting on the beam cross-section.

$$N_x = -\frac{E}{P}\iint(\eta\cos\Phi - \xi\sin\Phi)dA = 0$$

as $\iint_A \eta dA = 0$ & $\iint_A \xi dA = 0$

$$\int \eta dF_x = \iint_A \frac{-E}{P}(\eta^2\cos\Phi - \eta\xi\sin\Phi)dA = -M_\xi = -M$$

$$\left(\frac{E}{P}\cos\Phi\right)\iint_A \eta^2 dA - \left(\frac{E}{P}\sin\Phi\right)\iint_A \eta\xi dA = M$$

($\iint_A \eta\xi dA = 0$, η & ξ being principal axis)

$\left(\frac{E}{P}\cos\Phi\right)I_{\xi\xi} = M$, Also

$$\int \xi dA = = \iint_A \frac{-E}{P}(\eta\cos\Phi - \xi\sin\Phi)dA = M_\eta = 0$$

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$\rightarrow \left(\frac{E}{\rho}\text{Sin}\phi\right) I_{\eta\eta=0}$

$\rightarrow$ **$\text{Sin}\phi = 0$**

So, **$\phi = 0$**

Neutral Axis Coincides with axis about which moment is acting.

And, $\varepsilon_{xx} = - \frac{p}{\rho}$

$\rightarrow \varepsilon_{xx} = - \frac{(\eta \cos\phi - \xi \sin\phi)}{\rho}$

(Using $p = \eta\cos\phi - \xi\sin\phi$)

$\rightarrow \varepsilon_{xx} = - \frac{\eta}{\rho}$ $(\phi = 0)$

Now; $\rho = \frac{E}{M} \cos\phi \, I_{\xi\xi}$

$$\boxed{\varepsilon_{xx} = - \frac{\eta M}{E I_{\xi\xi}}}$$