LECTURE # 13 (PART II)

Q. Derivation of an Expression for an FM wave.
Assume the unmodulated carrier is represented as

$$V_c(t) = A_c \sin(\omega_c t + \varphi(t)) \quad -① $$

or

$$V_{FM}(t) = A_c \sin(\theta_i(t)) \quad -② .$$

where A_c is the unmodulated carrier amplitude
$\theta_i(t) = \omega_c(t) + \varphi(t)$ is the instantaneous angle of the FM modulated carrier.

In general the FM carrier can be represented by

$$\boxed{V_{FM} = A_c \sin[\theta_i(t)]} \quad -② .$$

This indicates that the carrier is unmodulated in Amplitude however the angle is modulated by the baseband modulating signal so that

$\theta_i(t)$ is the instantaneous value of the angle. If $\omega_i(t)$ is the instantaneous value of the angular frequency then

$$\boxed{\omega_i(t) = \frac{d}{dt}(\theta_i(t))} \quad -③$$

In other words $\omega_i(t)$ is given by the time rate of change of $\theta_i(t)$
We may also write $\boxed{\omega_i(t) = 2\pi f_i(t)}$ $-④$

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where $f_i(t)$ is the instantaneous value of the Frequency Modulated (FM) carrier.

From equation (3) we get

$$\boxed{\theta_i(t) = \int_0^t \omega_i(t)\,dt} \quad \text{--- (5)}$$

Substitute $\theta_i(t)$ from eqn (5) into eqn (2) we get

$$V_{FM} = A_c \sin\left[\int_0^t \omega_i(t)\,dt\right] \quad \text{--- (6)}$$

Using eqn (4) where $\omega_i(t) = 2\pi f_i(t)$ then eqn (6) can be written as

$$V_{FM} = A_c \sin\left[\int_0^t 2\pi (f_i(t))\,dt\right]$$

$$\text{or} \quad V_{FM} = A_c \sin\left[2\pi \int_0^t f_i(t)\,dt\right] \text{ - (7)}$$

From the Amp_time plot and Freq_time plot as depicted in previous lecture #12 on page 134, it is clear that in the FM wave the carrier frequency is not constant. Here f_c is the resting frequency and on FM implementation, the FM carrier is represented as the instantaneous frequency $f_i(t)$. Here FM carrier frequency changes around f_c in accordance to message $m(t)$ signal. Therefore we can write

$$\boxed{f_i(t) = f_c + K_f\, m(t)} \quad \text{--- (8)}$$

where f_c is the carrier resting frequency in Hz
k_f is frequency sensitivity in Hz/volt. It is a Constant.
$m(t)$ is message signal in volts.

Using Eqn (8), we can write Eqn (7) as

$$V_{FM} = A_c \sin\left[2\pi \int_0^t (f_c + k_f\, m(t))\, dt\right.$$

After integration we get

$$V_{FM} = A_c \sin\left[2\pi\left[(f_c t) + k_f \int_0^t m(t)\, dt\right]\right.$$

or $$V_{FM} = A_c \sin\left[(2\pi f_c t) + 2\pi k_f \int_0^t m(t)\, dt\right].$$

Since $\omega_c = 2\pi f_c$

$$\therefore \boxed{V_{FM} = A_c \sin\left[\omega_c t + 2\pi k_f \int_0^t m(t)\, dt\right]} \quad \text{--- (9)}$$

Eqn (9) represents the General expression of the FM carrier where the modulating signal $m(t)$ contained in the angle of the modulated carrier.

Similarly in PM (Phase Modulation) the phase directly changes in accordance to $m(t)$
If k_p is the phase sensitivity constant in units of radians per volt then

$$\boxed{V_{PM} = A_c \sin\left[\omega_c t + k_p\, m(t)\right]} \quad \text{--- (10)}$$

Eqn (10) represents the General expression of a PM carrier where phase is directly proportional to $m(t)$.

If $m(t) = 0$ the both Eqn (9) and Eqn (10) reduce to the form

$$V(t) = A_c \sin[\omega_c t]$$

This is the expression similar to Eqn (1) of an unmodulated carrier where phase is Zero. This implies that in absence of a modulating information signal, the output of FM and PM will simply be an un-modulated carrier.

Again refer to Eqn (8) ie

$$\boxed{f_i(t) = f_c + k_f\, m(t)} \quad \text{--- (8)}$$

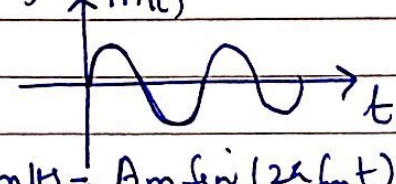
modulating voltage (volts)

Instantaneous Value of FM carrier (Hz)

Resting Frequency (Hz)

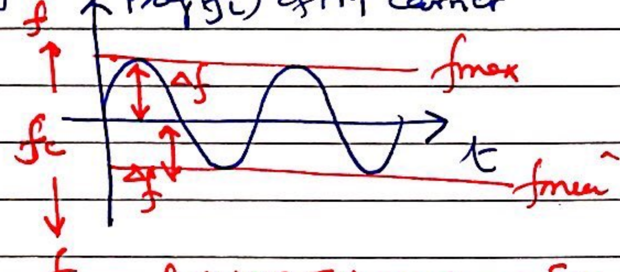
Frequency Sensitivity (Hz/Volt)

Pictorially Eqn (8) may be drawn as the freq-time plot as shown in Fig below



$m(t) = A_m \sin(2\pi f_m t)$

Modulating Signal



$f_i(t)$ of FM carrier – Freq-time Plot.

$$\therefore \quad f_{max} = f_c + \Delta f$$

or $\boxed{f_{max} = f_c + k_f A_m}$ (9.a) $\quad (\because \Delta f = k_f A_m$ by definition).

Similarly $f_{min} = f_c - \Delta f$

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or $\boxed{f_{min} = f_c - k_f A_m}$ — (9.b)

where A_m is peak amplitude of $m(t)$.

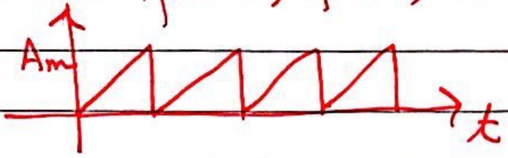
$\therefore$ Carrier swing about f_c is $= f_{max} - f_{min}$

$= (f_c + k_f A_m) - (f_c - k_f A_m)$.

$\therefore$ Carrier swing $= 2 k_f A_m$.

This applies typically for the $m(t)$ signal as given in this example.

Q: What will be the f_{max}, f_{min}, carrier swing for $m(t)$ as shown:-



Q. Use the general expression of FM carrier as given by Eqn (9) ie

$$V_{FM} = A_c \sin\left[\omega_c t + 2\pi k_f \int_0^t m(t)\,dt\right]$$

And draw the corresponding spectrum if

$m(t) = A_m \cos\omega_m t$.

$$V_{FM} = A_c \sin\left[\omega_c t + 2\pi k_f \int_0^t m(t)\,dt\right] \quad \text{— (9)}$$

Here $m(t) = A_m \cos\omega_m t$. Substitute $m(t)$ in above Eqn.

$$\therefore V_{FM} = A_c \sin\left[\omega_c t + 2\pi k_f \int_0^t (A_m \cos\omega_m t)\,dt\right].$$

After Integration

$$\text{or } V_{FM} = A_c \sin\left[\omega_c t + 2\pi k_f \left(\frac{A_m \sin\omega_m t}{\omega_m}\right)\right].$$

$$\text{or } V_{FM} = A_c \sin\left[\omega_c t + \cancel{2\pi} k_f \frac{(A_m \sin\omega_m t)}{\cancel{2\pi} f_m}\right]$$

$$\therefore V_{FM} = A_c \sin\left[\omega_c t + \frac{k_f A_m}{f_m} \sin \omega_m t\right].$$

We have already defined for FM carrier modulation index $\beta = \frac{\Delta f}{f_m}$ or $\beta = \frac{k_f A_m}{f_m}$.

$$\therefore V_{FM} = A_c \sin\left[\omega_c t + \beta \sin \omega_m t\right]$$

Since $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

$$\boxed{\therefore V_{FM} = A_c \sin(\omega_c t) \cos(\beta \sin \omega_m t) + A_c \cos(\omega_c t) \sin(\beta \sin \omega_m t)} \quad \text{--- (11)}$$

The above function is a complicated function as the expanded form involves angle $\sin(\omega_m t)$ in the argument of Cos and Sin. So we use Bessels function $J_n(\beta)$ where n is the order of Bessel function and β is modulation index as defined above.

Here

$\cos(\beta \sin \omega_m t)$, $\cos(\beta \sin \omega_m t) \rightarrow$ This is an even periodic function and its Fourier Series Expansion excludes all odd Harmonics

$\sin(\beta \sin \omega_m t)$. $\sin(\beta \sin \omega_m t) \rightarrow$ This is an odd periodic function and its Fourier Series expansion excludes all ~~odd~~ even Harmonics

$$\therefore \cos(\beta \sin \omega_m t) = J_0(\beta) + 2J_2(\beta)\cos(2\omega_m t) + 2J_4(\beta)\cos(4\omega_m t) + \text{———}$$

ie $J_0(\beta)$ + Even Harmonic.

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Similarly $\sin(\beta \sin \omega_m t) = 2J_1(\beta)\sin(\omega_m t) + 2J_3(\beta)\sin(3\omega_m t) + 2J_5 \sin(5\omega_m t) + \cdots$

ie all odd Harmonics

Substituting the Bessel's function expansion of $\cos(\beta \sin \omega_m t)$ & $\sin(\beta \sin \omega_m t)$ in Equation (II) we get.

$$
\begin{aligned}
V_{FM} = {} & A_c J_0(\beta)\sin(\omega_c t) + 2A_c J_1(\beta)\cos(\omega_c t)\sin(\omega_m t) \\
& + 2A_c J_2(\beta)\sin(\omega_c t)\cos(2\omega_m t) \\
& + 2A_c J_3(\beta)\cos(\omega_c t)\sin(3\omega_m t) \\
& + 2A_c J_4(\beta)\sin(\omega_c t)\cos(4\omega_m t) \\
& + 2A_c J_5(\beta)\cos(\omega_c t)\sin(5\omega_m t) \\
& + \cdots \text{ (till Infinity).}
\end{aligned}
$$

Since $2\cos A \sin B = \sin(A+B) - \sin(A-B)$

&

$2\sin A \cos B = \sin(A+B) + \sin(A-B)$

Carrier Compont | First Sideband.

$\therefore V_{FM} = A_c J_0(\beta)[\sin(\omega_c t)] + A_c J_1(\beta)[\sin(\omega_c+\omega_m)t - \sin(\omega_c-\omega_m)t]$

Second sideband $+ A_c J_2(\beta)[\sin(\omega_c+2\omega_m)t + \sin(\omega_c-2\omega_m)t]$

Third sideband $+ A_c J_3(\beta)[\sin(\omega_c+3\omega_m)t - \sin(\omega_c-3\omega_m)t]$

Fourth sideband $+ A_c J_4(\beta)[\sin(\omega_c+4\omega_m)t + \sin(\omega_c-4\omega_m)t]$

Fifth sideband $+ A_c J_5(\beta)[\sin(\omega_c+5\omega_m)t - \sin(\omega_c-5\omega_m)t]$

Sixth sideband $+ A_c J_6(\beta)[\sin(\omega_c+6\omega_m)t + \sin(\omega_c+6\omega_m)t]$

$+ \cdots$ $+ \cdots$ (till infinity)

Here in this form each Component can be described as follows:

$V_{FM} = A_c J_0(\beta)\left[\sin(\omega_c t)\right]$ → Carrier Component at $f = f_c$
→ Amplitude = $A_c J_0(\beta)$

+

$A_c J_1(\beta)\left[\sin(\omega_c+\omega_m)t - \sin(\omega_c-\omega_m)t\right]$ → First sideband at $f_c \pm f_m$
→ Amplitude = $A_c J_1(\beta)$.

+

$A_c J_2(\beta)\left[\sin(\omega_c+2\omega_m)t + \sin(\omega_c-2\omega_m)t\right]$ → Second Side Band at $f_c \pm 2f_m$
→ Amplitude = $A_c J_2(\beta)$

+

$A_c J_3(\beta)\left[\sin(\omega_c+3\omega_m)t - \sin(\omega_c-3\omega_m)t\right]$ → Third sideband at $f_c \pm 3f_m$
→ Amplitude = $A_c J_3(\beta)$.

+

$A_c J_4(\beta)\left[\sin(\omega_c+4\omega_m)t + \sin(\omega_c-4\omega_m)t\right]$ → Fourth Sideband at $f_c \pm 4f_m$
→ Amplitude = $A_c J_4(\beta)$.

+

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$A_c J_n(\beta)\left[\sin(\omega_c+n\omega_m)t \pm \sin(\omega_c-n\omega_m)t\right]$ → n^{th} Sideband at $f_c \pm nf_m$
→ Amplitude = $A_c J_n(\beta)$.

If n is odd then sign between two components is negative.
If n is even then sign between two components is positive.

Note that:-

- All components are frequency translated about or around ~~f_c by $f_c \pm nf_m$ and $f_c - f_m$ when~~, ~~around~~
- → f_c by $+nf_m$ and $-nf_m$.
- → This implies that Components in Spectrum will be at $\sin\omega_c t$, then at $f_c + nf_m$ and at $f_c - nf_m$.