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UNDERDAMPED MOTION OF GALVANOMETER.

We know here $D^2 < 4KJ$

$$m_1 = \frac{-D \pm \sqrt{D^2 - 4KJ}}{2J} = \frac{-D}{2J} + \frac{\sqrt{D^2 - 4KJ}}{2J}$$

$$= \frac{-D}{2J} + \sqrt{\frac{(4KJ - D^2)(-1)}{2J}}$$

$$= \frac{-D}{2J} + j\sqrt{\frac{4KJ - D^2}{2J}}$$

$m_1 = -\alpha + j\omega$; $m_2 = -\alpha - j\omega$

$\therefore$ From eq ②, we get

$$\theta = A e^{(-\alpha + j\omega)t} + B e^{(-\alpha - j\omega)t} + \theta_F$$

$e^{\pm j\omega t}$ is complex ; θ is real as it represents a physical quantity. Thus A & B must be complex.

Let $A = a + jb$; $B = c + jd$

$$\theta = e^{-\alpha t}\left[(a+jb)e^{j\omega t} + (c+jd) \cdot e^{-j\omega t}\right] + \theta_F$$

$$= e^{-\alpha t}\left[(a+jb)(\cos\omega t + j\sin\omega t) + (c+jd)(\cos\omega t - j\sin\omega t)\right] + \theta_F$$

$$= e^{-\alpha t}\left[(a+c)\cos\omega t - (b-d)\sin\omega t + j(b+d)\cos\omega t - j(a-c)\sin\omega t\right] + \theta_F \quad \text{--- (3)}$$

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Equating imaginary parts on both sides, hence for all values of t

$(b+d)\cos\omega t + (a-c)\sin\omega t = 0$

at $\omega t = 0$; $b+d = 0$ or $b = -d$

at $\omega t = \pi/2$; $a-c = 0$ or $a = c$.

$A = a+jb$ and $B = a-jb$

Thus A and B are complex conjugate pair

$\therefore$ From eq 3 ;

$\theta = 2e^{-\alpha t}[a\cos\omega t + d\sin\omega t] + \theta_F$ — 4

Let $a = F/2 \sin\phi$ $d = -F/2 \cos\phi$

$F = 2\sqrt{a^2+d^2}$ $\phi = \tan^{-1}(a/d)$

$\theta = Fe^{-\alpha t}[\sin\phi\cos\omega t + \cos\phi\sin\omega t] + \theta_F$

$= Fe^{-\alpha t}\sin(\omega t+\phi) + \theta_F$

$\theta = Fe^{-D/2J\, t}[\sin(\omega_d t+\phi)] + \theta_F$ — 5

where ω_d = angular frequency of damped oscillations, $\frac{\sqrt{4KJ-D^2}}{2J}$ rad/sec.

Let us evaluate θ_F and ϕ.

at $t=0$; $\theta = 0$.

$0 = F\sin\phi + \theta_F$ $\therefore$ $\boxed{\sin\phi = -\frac{\theta_F}{F}}$ — (x)

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Since θ_F and ϕ are +ve quantities, F is a negative quantity.

Differentiating eq (5), we get

$$\frac{d\theta}{dt} = \frac{-D}{2J} F e^{-D/2J\, t} \left[\sin(\omega_d t + \phi)\right]$$

$$+ F e^{-D/2J\, t} \omega_d \left[\cos(\omega_d t + \phi)\right]$$

at $t = 0$; $d\theta/dt = 0$

$$0 = \frac{-D}{2J} F \sin\phi + F\omega_d \cos(\phi) \quad \text{— (6)}$$

$$\tan\phi = \omega_d \frac{2J}{D}$$

$$= \frac{\sqrt{4KJ - D^2}}{2J} \cdot \frac{2J}{D} = \frac{\sqrt{4KJ - D^2}}{D}$$

From (6), we get

eq (X)

$$0 = \left(\frac{-D}{2J} \cdot F\right)\left(\frac{-\theta_F}{F}\right) + F\omega_d \cos\phi$$

$$\text{or} \quad \cos\phi = \frac{-D}{2J\omega_d} \cdot \frac{\theta_F}{F} \quad \text{— (7)}$$

From (X) & eq (7)

$$\sin^2\phi + \cos^2\phi = \left(\frac{-\theta_F}{F}\right)^2 + \left(\frac{-D}{2J\omega_d} \cdot \frac{\theta_F}{F}\right)^2$$

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$$\frac{\theta_F^2}{F^2} + \frac{D^2}{4J^2\omega_d^2 \cdot F^2} \cdot \theta_F^2 = 1$$

or $$\left(\frac{\theta_F}{F}\right)^2 \left[1 + \frac{D^2}{4J^2\omega_d^2}\right] = 1$$

$$\left(\frac{\theta_F}{F}\right)^2 \left[\frac{D^2 + 4J^2\omega_d^2}{4J^2\omega_d^2}\right] = 1$$

$$\frac{F^2}{\theta_F^2} = \frac{D^2 + 4J^2\omega_d^2}{4J^2\omega_d^2}$$

$$F = \frac{\pm \theta_F \sqrt{D^2 + 4J^2\omega_d^2}}{2J\omega_d}$$

Now since $\sin\phi = \frac{-\theta_F}{F}$;

ϕ is +ve, so is $\sin\phi$ and θ_F is also +ve

$\therefore$ F has to be negative.

$$F = \frac{-\theta_F \sqrt{D^2 + 4J^2\omega_d^2}}{2J\omega_d} \quad \text{——} \quad (8)$$

$$\omega_d = \frac{\sqrt{4KJ - D^2}}{2J}$$

$\therefore$ From eq (8), we get

$$F = \frac{-\theta_F \sqrt{D^2 + 4J^2\left(\frac{4KJ - D^2}{4J^2}\right)}}{2J \times \frac{\sqrt{4KJ - D^2}}{2J}}$$

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$$F = \frac{-\theta_F \sqrt{D^2 + 4KJ - D^2}}{\sqrt{4KJ - D^2}} = \frac{-2\theta_F \sqrt{KJ}}{\sqrt{4KJ - D^2}} \quad \text{—(9)}$$

From eq (5), we get

$$\theta = \frac{-2\theta_F \sqrt{KJ}}{\sqrt{4KJ - D^2}} e^{-D/2J\, t} \sin\left[\frac{\sqrt{4KJ-D^2}}{2J} t + \tan^{-1}\left(\frac{w_d/2J}{D}\right)\right] + \theta_F$$

$$\theta_F \left[1 - \frac{2\sqrt{KJ}}{\sqrt{4KJ-D^2}} \cdot e^{-D/2J\, t} \sin\left[\frac{\sqrt{4KJ-D^2}\, t}{2J} + \tan^{-1}\frac{\sqrt{4KJ-D^2}}{D}\right)\right] \quad \text{—(10)}$$

When a current is suddenly passed through the coil of an underdamped galvanometer, the moving system will start from its zero current position and then oscillate about its final steady state position θ_F. This oscillation would be an attentuated sinusoidal motion. The angular frequency of this sinusoidal component of motion is w_d.

The frequency of this sinusoidal component called frequency of damped oscillations.

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$$f_d = \frac{\omega_d}{2\pi} = \frac{1}{2\pi}\,\frac{\sqrt{4KJ-D^2}}{2J}$$

$$T_d = \frac{1}{f_d} = \frac{2\pi}{\omega_d} = 2\pi \cdot \frac{2J}{\sqrt{4KJ-D^2}}$$

(ii) UNDAMPED MOTION OF GALVANOMETER

For an undamped motion of galvanometer, the damping forces are equal to zero ($D=0$). Such a case is not possible under pratical working conditions but the properties of an undamped galvanometer are used in expressing its motion under actual ~~not~~ operating conditions

For $D=0$;

$$\omega_d = \omega_n$$

$$\omega_n = \frac{\sqrt{4KJ-D}}{2J} = \sqrt{\frac{K}{J}}$$

From eq (8), $F = -\theta_F \dfrac{\sqrt{0+4J^2\omega_d^2}}{2J\omega_d}$

$$F = -\theta_F \times \frac{2J\omega_d}{2J\omega_d} \quad \therefore \boxed{F = -\theta_F}$$

Frequency of undamped oscillation $f_n = \dfrac{\omega_n}{2\pi}$

$$= \frac{1}{2\pi}\sqrt{\frac{K}{J}}$$

Also $\tan\phi_0 = \dfrac{\sqrt{4KJ-D^2}}{D} = \infty$

$$\boxed{\phi_0 = 90^\circ}$$

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Substituting these values in eq ③,

$\theta = -\theta_F e^0 \sin(\omega nt + 90°) + \theta_F$

or $\boxed{\theta = \theta_F (1 - \cos \omega n t)}$

iii CRITICALLY DAMPED MOTION OF GALVANOMETER

For critically damped motion $D^2 = 4KJ$

Roots of aux. eq are $m_1, m_2 = -D/2J$.
& solution is

$$\theta = \theta_F + e^{-D/2J\, t} (A + Bt) \quad —⑪$$

The values of A and B are formed as
Differentiate wrt t.

$$\frac{d\theta}{dt} = \left\{ \frac{-D}{2J} e^{-D/2J\, t} [A + Bt] + B e^{-D/2J\, t} \right\}$$

$$= \left\{ e^{-D/2J\, t} \left[\frac{-D}{2J} (A + Bt) + B \right] \right.$$

at $t = 0$; $\theta = 0$ and $d\theta/dt = 0$

$0 = \theta_F + A$

$0 = \frac{-D}{2J} A + B$

$\therefore A = -\theta_F$ & $B = \frac{-D}{2J} \theta_F$

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Hence the solution is

$$\theta = \theta_F \left[1 - e^{-D/2J\, t} \left(1 + \frac{D}{2J} t \right) \right]$$

Under critical damping $D = D_c = 2\sqrt{KJ}$
Where D_c = damping constantant.

$$\frac{D}{2J} = \frac{2\sqrt{KJ}}{2J} = \sqrt{\frac{K}{J}} = \omega_n$$

∴ For a critically damped galvanometer,

$$\theta = \theta_F \left[1 - e^{-\omega_n t} (1 + \omega_n t) \right]$$

OPERATIONAL CONSTANTS

The user is interested in operational constants rather than intrinsic constants J, D & K. These constants are not known to user and their evaluation is difficult.

Op. Constants are :-
(i) Sensitivity (ii) Critical damping Resistance
(iii) Time period.

RELATIVE DAMPING

Damping is expressed most conveniently with critical damping case.
The damping ratio is defined as the ratio of actual damping constant to the damping constant required for critical damping.

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Damping ratio $\zeta = D/D_c$

But $D_c = 2\sqrt{KJ}$

$$\therefore \zeta = \frac{D}{2\sqrt{KJ}}$$

$$\frac{D}{2J} = \frac{D}{2\sqrt{KJ}} \times \sqrt{\frac{K}{J}} = \zeta \cdot \sqrt{\frac{K}{J}} = \zeta\, \omega_n$$

$$\tan\phi = \sqrt{\frac{4KJ - D^2}{D}} = \sqrt{\frac{4KJ}{D^2} - 1}$$

$$= \sqrt{\left(\frac{2\sqrt{KJ}}{D}\right)^2 - 1} = \sqrt{\frac{1}{\zeta^2} - 1} = \sqrt{\frac{1-\zeta^2}{\zeta}}$$

$$\sin\phi = \sqrt{1-\zeta^2}$$

$$\cos\phi = \zeta$$

We have $\dfrac{2\sqrt{KJ}}{\sqrt{4KJ - D^2}} = \sqrt{\dfrac{K}{J}} \cdot \dfrac{2J}{\sqrt{4KJ - D^2}} = \dfrac{\omega_n}{\omega_d}$

$$\frac{\omega_n}{\omega_d} = \frac{2\sqrt{KJ}}{\sqrt{4KJ - D^2}} = \frac{1}{\sqrt{1 - \frac{D^2}{4KJ}}}$$

But $D_c^2 = 4KJ$

$$\frac{\omega_n}{\omega_d} = \frac{1}{\sqrt{1 - \frac{D^2}{D_c^2}}} = \frac{1}{\sqrt{1-\zeta^2}}$$

$$\boxed{\omega_d = \omega_n \sqrt{1-\zeta^2}}$$

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Substituting all these values in eq (10), we get

$$\theta = \theta_F \left[1 - \frac{w_n}{w_d} e^{-\xi w_n t} \sin\left(w_d t + \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi} \right)\right)\right]$$

$$= \theta_F \left[1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi w_n t} \sin\left(w_d t + \tan^{-1} \frac{\sqrt{1-\xi^2}}{\xi} \right)\right] \longrightarrow (A)$$

Also $\frac{T_0}{T_d} = \frac{f_d}{f_n} = \frac{w_d}{w_n}$

But $\frac{w_d}{w_n} = \sqrt{1-\xi^2}$

$\therefore \frac{T_0}{T_d} = \sqrt{1-\xi^2}$

Also $\xi w_n = \frac{2\pi \xi}{T_0}$ $\therefore$ $\boxed{w_n = \frac{2\pi}{T_0}}$

From eq A, we get

$$\theta = \theta_F \left[1 - \frac{T_d}{T_0} e^{-\frac{2\pi \xi t}{T_0}} \sin\left(\frac{2\pi}{T_0}\sqrt{1-\xi^2}\, t + \sin^{-1}\sqrt{1-\xi^2} \right)\right]$$

$$\boxed{= \theta_F \left[1 - \frac{T_d}{T_0} e^{\left(-\frac{2\pi \xi t}{T_0}\right)} \cdot \sin\left(\frac{2\pi t}{T_d} + \sin^{-1} \frac{T_0}{T_d} \right)\right]} \longrightarrow (2)$$

$$\left(\because \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} = \sin^{-1}\sqrt{1-\xi^2} = \sin^{-1}\frac{T_0}{T_d} \right)$$

Eq (2) describes the galvanometer motion in terms of operational constants i.e. relative damping free period and sensitivity.