Lecture - 14

15/05/20

General strategy to prove that a certain Language is not regular

Def of a Regular Language

Regular Languages are languages accepted by DFA's, NFA's and ϵ-NFA's. These are defined by regular expressions.

Properties of Regular Languages

- Closure property — useful for building complex automata
- Decision Property — useful for finding if two automata define the same language $\Downarrow$ in turn useful for minimization of automata

Note

Whether a language is regular or not can be proved by use of Pumping Lemma.

Steps (To prove whether L is regular or not)

1. Assume that the language L is regular and n is the number of states of FA.

2. Select the string x such that $|x| \geq n$ and break it into substrings u, v and w so that with the constraints

$$x = uvw$$

$$v \neq \varepsilon \quad \text{i.e,} \quad |v| \geq 1$$

$$|uv| \leq n$$

3. Find any i such that uv^iw is not in L i.e, $uv^iw \notin L$.

According to pumping lemma, uv^iw is in L for $i \geq 0$.

So, the result is contradiction to the assumption that the language is regular.

Therefore, the given language is not regular.

Q1. Show that $L = \{ ww^R \mid w \in (0+1)^* \}$ (3)
is not regular.

Sol.

Step 1 :→ Let L be regular and n be the number of states in the FA.

Consider the string:

$$x = \overbrace{\underbrace{1\cdots1}_{n}\ \underbrace{0\cdots0}_{n}}^{w}\ \overbrace{\underbrace{0\cdots0}_{n}\underbrace{1\cdots1}_{n}}^{w^R}$$

$$w = 1\cdots10\cdots0$$

$$w^R = 0\cdots01\cdots1$$

step 2 :→

Since $|x| \geq n$, we can split the string x into uvw such that $|uv| \leq n$ and $|v| \geq 1$ as under.

$$x = \underbrace{1\cdots\cdots}_{u}\underbrace{1}_{v}\underbrace{0\cdots0\,0\cdots01\cdots1}_{w}$$

$|u| = n-1$

$|v| = 1$ so that $|u| + |v| = n-1+1 = n$

which is true.

According to Pumping Lemma, $uv^iw \in L$ for $i = 0, 1, 2, \ldots$

Step 3

If i is 0 i.e, v does not appear and so the number of 1's on the left of x will be less than the number of 1's on the right of x, hence the string is not of the form ww^R.

So $uv^iw \notin L$ when $i = 0$.

This is a contradiction to the assumption that the language is regular.

Hence $L = \{ww^R \mid w \in (0+1)^*\}$ is not regular

Q2) Show that $L = \{a^i b^j \mid i > j\}$ is not regular.

Sol

Step 1 :→

Let L be regular and n be the number of states in FA.

Consider the string $x = a^{n+1} b^n$

Step 2 $|x| = 2n+1 \geq n$, we can split x into uvw.

$|uv| \leq n$ and $|v| \geq 1$

$$x = a^{n+1} b^n = \underset{u}{a^j} \, \underset{v}{a^k} \, \underset{w}{ab^n}$$

$|u| = j$ and $|v| = k \geq 1$,

$\therefore \quad |uv| = |u| + |v| = j+k \leq n$

Step 3 According to Pumping Lemma,
$uv^i w \in L$ for $i \geq 0$

i.e, $a^j (a^k)^i ab^n \in L$ for $i \geq 0$

if $i = 0$, number of a's in string u will not be more than the number of b's. in w, which is a contradiction, to the assumption that number of a's are more than the number of b's.

So, the language $L = \{a^i b^j \mid i > j\}$ is not regular

Q) Show that $L = \{a^{n!} \mid n \geq 0\}$ is not regular.

Sol

(1) Let L be regular and n the number of states in the FA.

(2) Let $x = a^{n!} \Rightarrow |x| \geq n$.

Split $x = uvw$ such that $|uv| \leq n$ and $|v| \geq 1$.

$$x = a^{n!} = \underset{u}{a^{j}}\ \underset{v}{a^{k}}\ \underset{w}{a^{n!-j-k}}$$

$$|u| = j \qquad |v| = k \geq 1$$

$$\therefore |uv| = |u| + |v| = j+k \leq n$$

(3) According to Pumping Lemma,

$$uv^{i}w \in L \text{ for } i = 0, 1, 2, \ldots$$

i.e., $a^{j} (a^{k})^{i} a^{n!-j-k} \in L$

if $i = 0 \Rightarrow a^{j} a^{n!-j-k} \in L$

i.e., $uw \in L$

$$\Rightarrow a^{n!-k} \in L.$$

$\therefore n! > n! - k$ — if $k = 1$

$n! > n! - 1$ (correct in general)

but $n! = n! - 1 \Rightarrow$ according to Pumping Lemma

L is NOT regular $\Leftarrow$ contradiction $\Leftarrow$

Q/ Prove $L = \{ w \mid n_a(w) < n_b(w) \}$ is not regular. (7)

Sol: (1) Let L be regular and n the number of states in the FA.

Let $x = a^{n-1} b^n \in L$

(2) $|x| = 2n-1 \geq n$

$x = a^{n-1} b^n$

$= a^{n-1} \; b^k \; b^{n-k} = uvw$

$|u| = n-1 \quad |v| = k$.

$|uv| = |u| + |v| = n-1+k \leq n$

To satisfy this condition, the value of k should be less than or equal to 1.

(3) According to Pumping Lemma, $uv^i w \in L$ for $i = 0, 1, 2 \ldots$

$a^{n-1} (b^k)^i \, b^{n-k} \in L$ for $i = 0, 1, 2, \ldots$

if $i = 0$

$a^{n-1} \, b^{n-k} \in L$

if $k = 1$

$a^{n-1} \, b^{n-1} \in L \Rightarrow$ no. of a's = no of b's

$\Downarrow$

L is not regular. $\Longleftarrow$ contradiction.

Q) Show that

$L = \{ a^n \mid n = k^2 \text{ for } k \geq 0 \}$

is not regular.

Sol.

(1) L is regular, n is number of states in FA.

$x = a^m \in L \quad (m = n^2)$

i.e, we chose a value of x which depends on n.

(2) $|x| \geq n$

$x = a^m = \underset{u}{a^j}\ \underset{v}{a^k}\ \underset{w}{a^{m-j-k}}$

$|u| = j \qquad |v| = k \geq 1$

$|uv| = |u| + |v| = j + k \leq n$

(3) According to Pumping Lemma,

$uv^iw \in L$ for $i = 0, 1, 2, \ldots$

i.e, $a^j a^{ki} a^{m-j-k} \in L$ for $i = 0, 1, 2$

if $i = 2$

$a^j a^{2k} a^{m-j-k} \in L$

$a^{k+m} \in L \qquad k \geq 1$

$|a^{k+m}| = m + k = n^2 + k$

$\therefore n^2 < n^2 + k < n^2 + 1 < n^2 + 2n + 1$

~~$n^2 < (n+1)^2$~~

Contradiction

NOT regular ← $n^2 < n^2 + k < n^2 + 2n + 1$

not a perfect square