Lectures 16,17,18,19,20 + tutorial No-2

LINEAR, NONLINEAR AND DYNAMIC PROGRAMMING

The basic techniques used in water resources systems analysis are optimization and simulation. Whereas optimization techniques are meant to give global optimum solutions, simulation is a trial and error approach leading to the identification of the best solution possible. The simulation technique cannot guarantee global optimum solution; however, solutions, which are very close to the optimum, can be arrived at using simulation and sensitivity analysis. Optimization models are embodied in the general theory of mathematical programming. They are characterized by a mathematical statement of the ob jective function, and a formal search procedure to determine the values of decision variables, for optimizing the objective function. The principal optimization techniques are:

1. ***Linear Programming***

The objective function and the constraints are all linear. It is probably the single-most applied optimization technique the world over. In integer programming, which is a variant of linear programming, the decision variables take on integer values. In mixed integer programming, only some of the variables are integer

2.*Nonlinear Programming*

The objective function and/or (any of) the constraints involve nonlinear terms. General solution procedures do not exist. Special purpose solutions, such as quadratic programming, are available for limited applications. However, linear programming may still be used in some engineer-ing applications, if a nonlinear function can be either transformed to a linear function, or approximated by piece-wise linear segments.

3. ***Dynamic Programming***

Offers a solution procedure for linear or nonlinear problems, in which multistage decision-making is involved. The choice of technique for a given problem depends on the configuration of the system being analyzed, the nature of the objective function and the constraints, the availability and reliability of data, and the depth of detail needed in the investigation. Linear programming (LP), and dynamic programming (DP)are the most common mathematical programming models used in water re-sources systems analysis. Simulation, by itself, or in combination with LP, DP,or both LP and DP is used to analyze complex water resources systems

Discounting refers to translating future money to its present value, and com-pounding refers to translating present money to a future period in time. For equivalence calculation of money at different points of time and for comparison of engineering alternatives, we need to know the commonly used discounting factors: The following notation is used in this section.

i= interest rate per year expressed in per cent

P= present sum of money

F = future sum of money

A= annual payment, or payment per period Usually, three values out of these four would be known and it will be required to compute the fourth

Example 3.1.2 Suppose we buy a pump for Rs 10,000 today. It has a service life of 10 years with no salvage value at the end of its service life. We take a loan of Rs 10,000 from the bank at 6% interest rate.

The amount required to be paid at the end of each year to the bank to repay the loan completely in 10 years with interest is

A = 10,000 × Capital Recovery Factor, $[A/P, 6\%, 10]$, with $i = 6$ and $n = 10$.

= 10,000 × 0.13587 = Rs 1,358.7

The loan of Rs 10,000 will be repaid with interest if Rs 1,358.7 is paid to the bank at the end of each year for 10 years.

On the other hand, we bought the pump for Rs 10,000 (borrowed from the bank) and it is in our hands today (but we have no other money). Every year, we raise a fund and, at the end of the year, deposit in the bank to have enough money at the end of 10 years to buy another new pump. Assuming that the cost of the pump remains the same, and that money grows at the same interest rate i (as when we borrowed the money to buy the pump), the annual sinking fund is

A' = 10,000 × sinking fund factor $[A'/F, 6\%, 10]$ with $i = 6$ and $n = 10$

= 10,000 × 0.07587 = Rs 758.7 (This is the monetary equivalent of the annual depreciation of the pump).

With this annual payment, we will have Rs 10,000 at the end of 10 years in the bank to buy a new pump. Only then, at the end of 10 years we will be in the same position as we are now. But remember, we should also pay the bank interest annually on the borrowed money of Rs 10,000 at 6% per year. This annual interest amount is

Annual interest = Pi

= 10,000 × .06 = Rs 600

which is the exact difference between the amounts A and A',

$(A - A' = 600)$,

i.e. Amount of capital recovery = Interest + Sinking fund (annual)

Q. Of the two plans given below, find the most feasible one by various methods.

	Plan A	*Plan B*
Cost of Equipment	50,000	35,000
Annual O & M Costs	2,000	2,500
Salvage value	7,000	6,000
Service life	30 years	15 years

Let us compare the two alternate plans by different methods

1. Equivalent Annual Cost (AC)

Plan A

Annual Cost = Interest + Depreciation + O & M costs

Interest on borrowed capital = 50,000 × .06 = 3000

$$\text{Depreciation} = (P - L)\,(A'/F,\ 6\%,\ 30)$$
$$= (50{,}000 - 7{,}000)\,(.01265) = 543.95$$

O & M Costs = 2000

$$\text{Total } AC = 3000 + 543.95 + 2000 = 5543.95$$

Plan B

Interest = 35,000 × .06 = 2100

$$\text{Depreciation} = (35{,}000 - 6{,}000)\left(\frac{A'}{F}, 6\%, 15\right)$$
$$= 29{,}000 \times (0.04296)$$
$$= 1245.84$$

O & M Cost = 2500

$$\text{Total } AC = 2100 + 1245.84 + 2500$$
$$= 5845.84$$

Plan *A* is preferable because of lower annual cost.

2. Present Worth (PW) Comparison

As mentioned earlier, PW should be compared for equal periods of analysis in both plans. Let us compare the plans on a 30-year basis. For this purpose, the life of plan *B* is extended by another 15 years with identical expenditure pattern as in the first 15 years. A comparison of present worth of all costs can be made then, based on an analysis period of 30 years.

Plan A (30 years)

PW of initial cost 50,000 = Rs 50,000

PW of salvage value = –7000 (P/F, 6%, 30)

(negative cost) = –7000 (0.17411) = –1218.77

PW of O & M Costs = 2000 (P/A, 6%, 30)

= 2000 (13.7648) = 27,529.60

PW of total cost = 50,000 – 1218.77 + 27,529.60

= 76,310.83

Plan B (for 30 years)

PW of initial cost of 35,000 = 35,000

PW of 35,000 at 15 years = 35,000 (P/F, 6%, 15)

= 35,000 (.41727) = 14604.45

PW of 6000 (negative cost at 15 years)

= –6000 (P/F, 6%, 15)

= –6000 (.41727) = –2503.62

PW of 6000 (negative cost at 30 years)

= –6000 (P/F, 6%, 30)

= –6000(.17411) = –1044.66

PW of O & M Costs = 2500 (P/A, 6%, 30)

= 2500 (13.7648) = 34,412

PW of Total Costs

= 35,000 + 14,604.45 – 2503.62 – 1044.66 + 34,412

= 80467.72

Because of lower costs Plan *A* is preferred to Plan *B*.

OPTIMIZATION AND SIMULATION

INTRODUCTION

In the previous lecture we studied the basics of an optimization problem and its formulation as a mathematical programming problem. In this lecture we look at the various criteria for classification of optimization problems, economic considerations and challenges in water resources.

MODELLING TECHNIQUES

The modelling or system analysis techniques were developed during the Second World War to deploy limited resources in an optimum manner. Since, these techniques were aided for military operations, these were known as operation research techniques. The popular operations research techniques include optimization methods, simulation, game theory, queuing theory etc. Among, these, the popular ones in water resources field are optimization and simulation.

OPTIMIZATION

Optimization is the science of choosing the best amongst a number of possible alternatives. There may be number of possible solutions for many engineering problems. It is required to identify the best through evaluation. The driving force in the optimization is the objective function (or functions). The optimal solution is the one which gives the best (either maximum or minimum) solution under all assumptions and constraints. Optimization theory is defined as the branch of mathematics dealing with techniques for maximizing or minimizing an objective function subject to linear, non-linear and integer constraints.

An optimization model can be stated as:

Objective function: Maximize (or Minimize) $f(X)$

Subject to the constraints $g_j(X) \geq 0, \quad j = 1,2,..,m$

$$h_j(X) = 0, \quad j = m+1, m+2,.., p$$

where X is the vector of decision variables, $g(X)$ are the inequality constraints and $h(X)$ are the equality constraints.

Classification of Optimization Techniques

Optimization problems can be classified based on the type of constraints, nature of design variables, physical structure of the problem, nature of the equations involved, permissible value of the design variables, deterministic/ stochastic nature of the variables, separability of the functions and number of objective functions. These methods are briefly discussed below.

(1)Classification based on existence of constraints

Under this category optimization problems can be classified into two groups as follows: Constrained optimization problems: which are subject to one or more constraints. Unconstrained optimization problems: in which no constraints exist.

(2)Classification based on the physical structure of the problem

Based on the physical structure, optimization problems are classified as optimal control and non-optimal control problems. (i) Optimal control problems An *Optimal control* (OC) problem is a mathematical programming problem involving a number of stages, where each stage evolves from the preceding stage in a prescribed manner. It is defined by two types of variables: the control or design and state variables. The *control variables* define the system and controls how one stage evolves into the next. The *state variables* describe the behavior or status of the system at any stage. The problem is to find a set of control variables such that the total objective function (also known as the performance index, PI) over all stages is minimized, subject to a set of constraints on the control and state variables. An OC problem can be stated as follows:

$$\text{Find } \mathbf{X} \text{ which minimizes } f(\mathbf{X}) = \sum_{i=1}^{l} f_i(x_i, y_i)$$

Subject to the constraints

$$q_i(x_i, y_i) + y_i = y_{i+1} \qquad i = 1, 2, \ldots, l$$

$$g_j(x_j) \le 0, \qquad j = 1, 2, \ldots, l$$

$$h_k(y_k) \le 0, \qquad k = 1, 2, \ldots, l$$

Where x_i is the ith control variable, y_i is the ith state variable, and f_i is the contribution of the ith stage to the total objective function. g_j, h_k, and q_i are the functions of x_j, y_j ; x_k, y_k and x_i, y_i respectively, and l is the total number of states. The control and state variables x_i and y_i can be vectors in some cases.

(3)Classification based on the nature of the equations involved

Based on the nature of equations for the objective function and the constraints, optimization problems can be classified as linear, non-linear, geometric or quadratic programming problems. This classification is much useful from a computational point of view since many predefined special methods are available for effective solution of a particular type of problem.

(4) Classification based on the permissible values of the decision variables

Under this classification, objective functions can be classified as integer and real-valued programming problems. (i) Integer programming problem
If some or all of the design variables of an optimization problem are restricted to take only integer (or discrete) values, the problem is called an *integer programming problem.* For example, the optimization problem is to find number of articles needed for an operation with least effort. Thus, minimization of the effort required for the operation being the objective, the decision variables, i.e., the number of articles used can take only integer values. Other restrictions on minimum and maximum number

of usable resources may be imposed. (ii) Real-valued programming problem A real-valued problem is that in which it is sought to minimize (or maximize) a real function by systematically choosing the values of real variables from within an allowed set. When the allowed set contains only real values, it is called a real-valued programming problem.

(5) Classification based on deterministic/ stochastic nature of the variables

Under this classification, optimization problems can be classified as deterministic or stochastic programming problems. (i) Deterministic programming problem In a deterministic system, for the same input, the system will produce the same output always. In this type of problems all the design variables are deterministic.
(ii) Stochastic programming problem In this type of an optimization problem, some or all the design variables are expressed probabilistically (non-deterministic or stochastic).

(6) Classification based on the number of objective functions

Under this classification, objective functions can be classified as single-objective and multi-objective programming problems. (i) Single-objective programming problem in which there is only one objective function. (ii) Multi-objective programming problem

OPTIMISATION BY LINEAR PROGRAMMING

Linear Programming (LP)
It is the most useful optimization technique used for the solution of engineering problems. The term „linear" implies that the objective function and constraints are „linear" functions of „nonnegative" decision variables. Thus, the conditions of LP problems (LPP) are

1. Objective function must be a linear function of decision variables
2. Constraints should be linear function of decision variables
3. All the decision variables must be nonnegative

For example,

Maximize	$Z = 6x + 5y$	$\prec$ Objective Function
subject to	$2x - 3y \leq 5$	$\prec$ 1st Constraint
	$x + 3y \leq 11$	$\prec$ 2nd Constraint
	$4x + y \leq 15$	$\prec$ 3rd Constraint
	$x,\ y \geq 0$	$\prec$ Nonnegativ ity Condition

is an example of LP problem. However, example shown above is in "general" form.

STANDARD FORM OF LPP

Standard form of LPP must have following three characteristics:

1. Objective function should be of maximization type
2. All the constraints should be of equality type
3. All the decision variables should be nonnegative

The procedure to transform a general form of a LPP to its standard form is discussed below. Let us consider the following example.

$$\begin{aligned} \text{Minimize} \quad & Z = -3x_1 - 5x_2 \\ \text{subject to} \quad & 2x_1 - 3x_2 \le 15 \\ & x_1 + x_2 \le 3 \\ & 4x_1 + x_2 \ge 2 \\ & x_1 \ge 0 \\ & x_2 \text{ unrestricted} \end{aligned}$$

The above LPP is violating the following criteria of standard form:

1. Objective function is of minimization type
2. Constraints are of inequality type
3. Decision variable is unrestricted, i.e., it can take negative values also, thus violating the non-negativity criterion. 2 x

However, a standard form for this LPP can be obtained by transforming it as follows: Objective function can be rewritten as

$$\text{Maximize} \quad Z' = -Z = 3x_1 + 5x_2$$

The first constraint can be rewritten as: $2x_1 - 3x_2 + x_3 = 15$. Note that, a new nonnegative variable x_3 is added to the left-hand-side (LHS) to make both sides equal. Similarly, the second constraint can be rewritten as: $x_1 + x_2 + x_4 = 3$. The variables x_3 and x_4 are known as *slack variables*. The third constraint can be rewritten as: $4x_1 + x_2 - x_5 = 2$. Again, note that a new nonnegative variable x_5 is subtracted form the LHS to make both sides equal. The variable x_5 is known as *surplus variable*.

Decision variable x_2 can expressed by introducing two extra nonnegative variables as

$$x_2 = x_2' - x_2''$$

Thus, x_2 can be negative if $x_2' < x_2''$ and positive if $x_2' > x_2''$ depending on the values of x_2' and x_2''. x_2 can be zero also if $x_2' = x_2''$.

After obtaining solution for x_2' and x_2'', solution for x_2 can be obtained as, $x_2 = x_2' - x_2''$.

CANONICAL FORM OF LPP

Canonical form of standard LPP is a set of equations consisting of the „objective function" and all the „equality constraints" (standard form of LPP) expressed in *canonical form*. Understanding the canonical form of LPP is necessary for studying *simplex method*, the most popular method of solving LPP.

APPLICATION OF LINEAR PROBLEM IN WATER RESOURCES

Q (1) Consider two crops 1 and 2. One unit of crop 1 brings four units of profit and one unit of crop 2 brings five units of profit. The demand of production of crop 1 is A units and that of crop 2 is B units. Let *x* be the amount of water required for A units of crop 1 and *y* be the same for B units of crop 2. The linear relations between the amounts of crop produced (i.e., demands A and B) and the available water (i.e., x and y) for two crops are shown below.

A = 0.5(x - 2) + 2

B = 0.6(y - 3) + 3

Minimum amount of water that must be provided to 1 and 2 to meet their demand is two and three units respectively. Maximum availability of water is ten units. Find out the optimum pattern of irrigation.

Solution: The objective is to maximize the profit from crop 1 and 2, which can be represented as *Maximize* $f = 4A + 5B$

Expressing as a function of the amount of water,

Maximize f = 4[0.5(x - 2) + 2] + 5[0.6(y - 3) + 3]= 2x + 3y + 10

subject to ; x+y <= Maximum availability of water ;

x.=2 ; Minimum amount of water required for crop 1 ;

y.=3:Minimum amount of water required for crop 2

The above problem is same as maximizing $f' = 2x + 3y$ subject to same constraints.

Changing the problem into standard form by introducing slack variables *S1, S2, S3*

Maximize f' = 2x + 3y

subject to

x + y + S1 =10

-x + S2 = -2

-y + S3 = -3

This problem is solved by forming the simplex table as:-

Starting Solution:

Basic Variables	Variables					RHS	Ratio
	x	y	S_1	S_2	S_3		
f'	-2	-3	0	0	0	0	
S_1	1	1	1	0	0	10	10
S_2	-1	0	0	1	0	-2	-
S_3	0	-1	0	0	1	-3	3

Iteration 1:

Basic Variables	Variables					RHS	Ratio
	x	y	S_1	S_2	S_3		
f'	-2	0	0	0	-3	9	-
S_1	1	0	1	0	1	7	7
S_2	-1	0	0	1	0	-2	2
y	0	1	0	0	-1	3	-

Iteration 2:

Basic Variables	Variables					RHS	Ratio
	x	y	S_1	S_2	S_3		
f'	0	0	0	-2	-3	13	-
S_1	0	0	1	1	1	5	5
x	1	0	0	-1	0	2	-
y	0	1	0	0	-1	3	-3

Iteration 3:

Basic Variables	Variables					RHS	Ratio
	x	y	S_1	S_2	S_3		
f'	0	0	3	1	0	28	-
S_3	0	0	1	1	1	5	-
x	1	0	0	-1	0	2	-
y	0	1	1	1	0	8	-

Hence the solution is

$x = 2$; $y = 8$; $f' = 28$

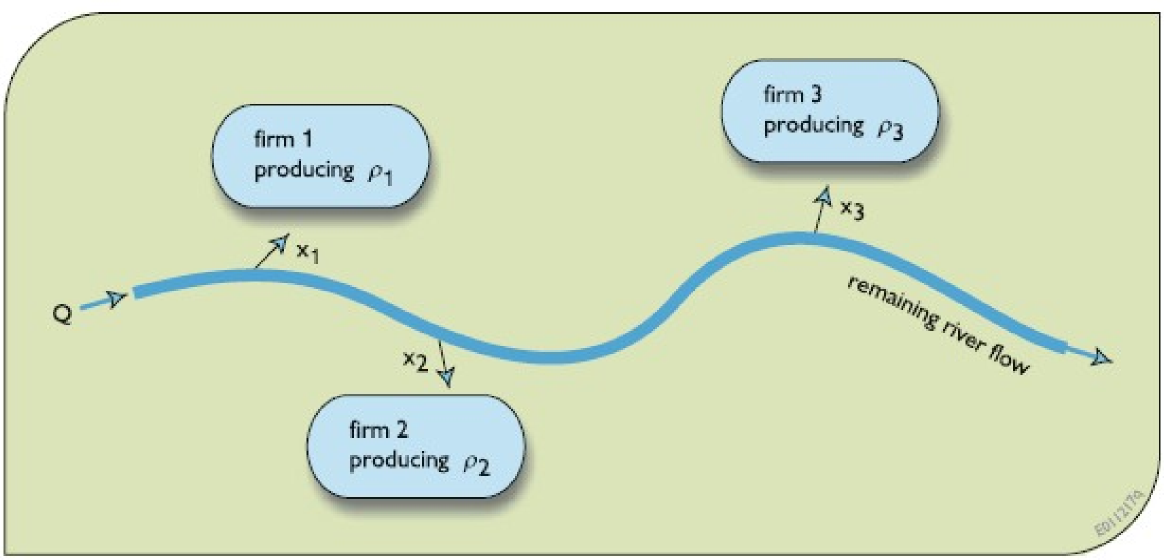
Therefore, $f = 28+10 = 38$

Thus, water allocated to crop A is 2 units and to crop B is 8 units and total profit yielded is 38 units.

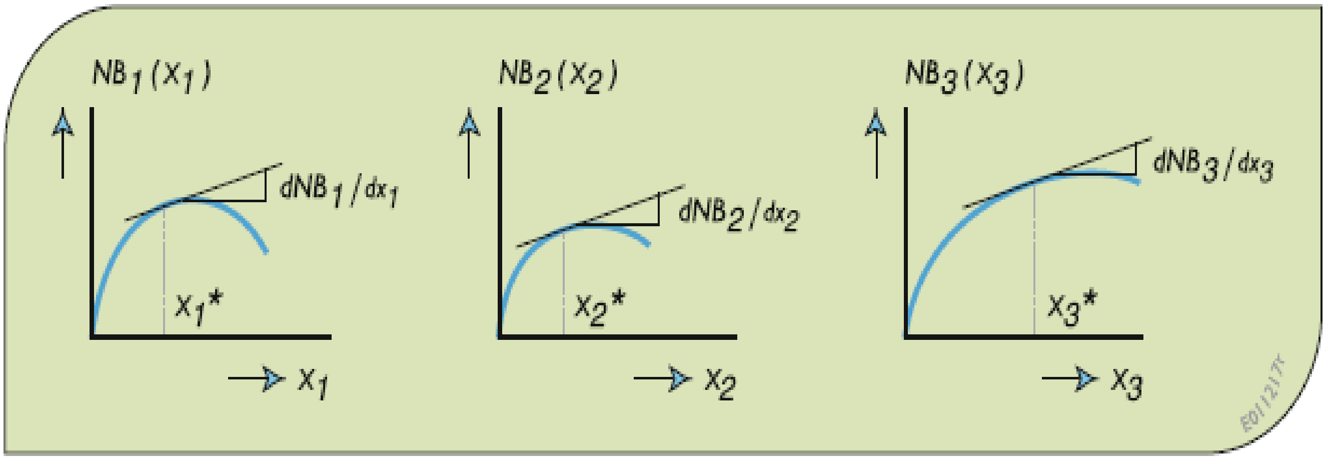
Nonlinear Optimization Models and Solution Procedures

Constrained optimization involves finding the values of decision variables given specified relationships that have to be satisfied. Constrained optimization is also called mathematical programming. Mathematical programming techniques include calculus-based Lagrange multipliers and various methods for solving linear and nonlinear models including dynamic programming, quadratic programming, fractional programming, and geometric programming, to mention a few. The applicability of each of these as well as other constrained optimization procedures is highly dependent on the mathematical structure of the model that in turn is dependent on the system being analyzed. Individuals tend to construct models in a way that will allow them to use a particular optimization technique they think is best. Thus, it pays to be familiar with various types of optimization methods since no one method is best for all optimization problems. Each has its strengths and limitations.

Consider a river from which diversions are made to three water-consuming firms that belong to the same corporation. Each firm makes a product. Water is needed in the process of making that product, and is the critical resource. The three firms can be denoted by the index j = 1, 2, and 3 and their water allocations by xj. Assume the problem is to determine the allocations xj of water to each of three firms (j = 1, 2, 3) that maximize the total net benefits, $\sum_j NB_j(x)_j$, obtained from all three firms. The total amount of water available is constrained or limited to a quantity of Q.



Three water-using firms obtain water from a river. The amounts xj allocated to each firm j will depend on the river flow Q



Net benefit functions for three water users, j, and their slopes at allocations xj

Assume the net benefits, NBj(xj), derived from water xj allocated to each firm j, are defined by:-

$$NB_1(x_1) = 6x_1 - x_1^2$$

$$NB_2(x_2) = 7x_2 - 1.5x_2^2$$

$$NB_3(x_3) = 8x_3 - 0.5x_3^2$$

These are concave functions exhibiting decreasing marginal net benefits with increasing allocations. These functions look like hills, as illustrated in Fig. above.